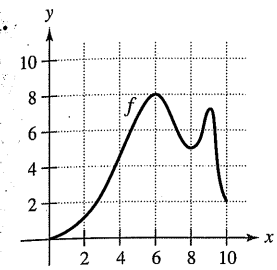
# 3.3 Exercises

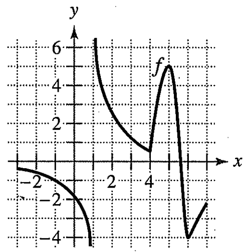
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

**Using a Graph** **In Exercises 1 and 2, use the graph of $f$ to find (a) the largest open interval on which $f$ is increasing, and (b) the largest open interval on which $f$ is decreasing.**

**1.**

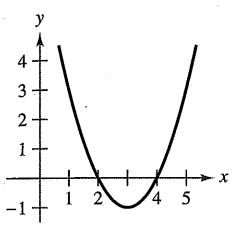



**2.**

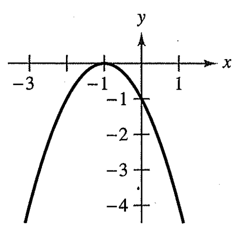



**Using a Graph** **In Exercises 3–8, use the graph to estimate the open intervals on which the function is increasing or decreasing. Then find the open intervals analytically.**

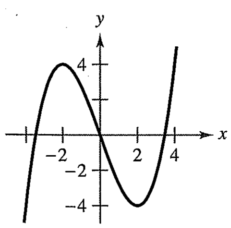
**3.** $f(x) = x^2 - 6x + 8$



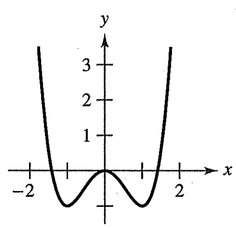

**4.** $y = -(x + 1)^2$



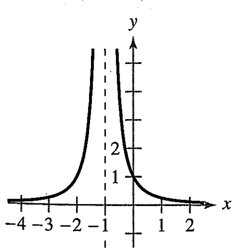

**5.** $y = \dfrac{x^3}{4} - 3x$



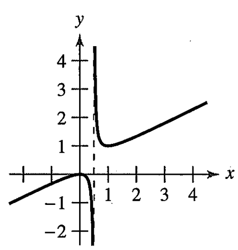

**6.** $f(x) = x^4 - 2x^2$




**7.** $f(x) = \dfrac{1}{(x + 1)^2}$




**8.** $y = \dfrac{x^2}{2x - 1}$




**Intervals on Which $f$ Is Increasing or Decreasing** **In Exercises 9–16, identify the open intervals on which the function is increasing or decreasing.**

**9.** $g(x) = x^2 - 2x - 8$

**10.** $h(x) = 12x - x^3$

**11.** $y = x\sqrt{16 - x^2}$

**12.** $y = x + \dfrac{9}{x}$

**13.** $f(x) = \sin x - 1, \quad 0 < x < 2\pi$

**14.** $h(x) = \cos\dfrac{x}{2}, \quad 0 < x < 2\pi$

**15.** $y = x - 2\cos x, \quad 0 < x < 2\pi$

**16.** $f(x) = \sin^2 x + \sin x, \quad 0 < x < 2\pi$

**Applying the First Derivative Test** **In Exercises 17–40, (a) find the critical numbers of $f$ (if any), (b) find the open interval(s) on which the function is increasing or decreasing, (c) apply the First Derivative Test to identify all relative extrema, and (d) use a graphing utility to confirm your results.**

**17.** $f(x) = x^2 - 4x$

**18.** $f(x) = x^2 + 6x + 10$

**19.** $f(x) = -2x^2 + 4x + 3$

**20.** $f(x) = -3x^2 - 4x - 2$

**21.** $f(x) = 2x^3 + 3x^2 - 12x$

**22.** $f(x) = x^3 - 6x^2 + 15$

**23.** $f(x) = (x - 1)^2(x + 3)$

**24.** $f(x) = (x + 2)^2(x - 1)$

**25.** $f(x) = \dfrac{x^5 - 5x}{5}$

**26.** $f(x) = x^4 - 32x + 4$

**27.** $f(x) = x^{1/3} + 1$

**28.** $f(x) = x^{2/3} - 4$

**29.** $f(x) = (x + 2)^{2/3}$

**30.** $f(x) = (x - 3)^{1/3}$

**31.** $f(x) = 5 - |x - 5|$

**32.** $f(x) = |x + 3| - 1$

**33.** $f(x) = 2x + \dfrac{1}{x}$

**34.** $f(x) = \dfrac{x}{x - 5}$

**35.** $f(x) = \dfrac{x^2}{x^2 - 9}$

**36.** $f(x) = \dfrac{x^2 - 2x + 1}{x + 1}$

**37.** $f(x) = \begin{cases} 4 - x^2, & x \le 0 \\ -2x, & x > 0 \end{cases}$

**38.** $f(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 - 2, & x > -1 \end{cases}$

**39.** $f(x) = \begin{cases} 3x + 1, & x \le 1 \\ 5 - x^2, & x > 1 \end{cases}$

**40.** $f(x) = \begin{cases} -x^3 + 1, & x \le 0 \\ -x^2 + 2x, & x > 0 \end{cases}$

**Applying the First Derivative Test** **In Exercises 41–48, consider the function on the interval $(0, 2\pi)$. For each function, (a) find the open interval(s) on which the function is increasing or decreasing, (b) apply the First Derivative Test to identify all relative extrema, and (c) use a graphing utility to confirm your results.**

**41.** $f(x) = \dfrac{x}{2} + \cos x$

**42.** $f(x) = \sin x \cos x + 5$

**43.** $f(x) = \sin x + \cos x$

**44.** $f(x) = x + 2\sin x$

**45.** $f(x) = \cos^2(2x)$

**46.** $f(x) = \sin x - \sqrt{3}\cos x$

**47.** $f(x) = \sin^2 x + \sin x$

**48.** $f(x) = \dfrac{\sin x}{1 + \cos^2 x}$

### Finding and Analyzing Derivatives Using Technology

**In Exercises 49–54, (a) use a computer algebra system to differentiate the function, (b) sketch the graphs of $f$ and $f'$ on the same set of coordinate axes over the given interval, (c) find the critical numbers of $f$ in the open interval, and (d) find the interval(s) on which $f'$ is positive and the interval(s) on which it is negative. Compare the behavior of $f$ and the sign of $f'$.**

**49.** $f(x) = 2x\sqrt{9 - x^2}, \quad [-3, 3]$

**50.** $f(x) = 10\left(5 - \sqrt{x^2 - 3x + 16}\right), \quad [0, 5]$

**51.** $f(t) = t^2 \sin t, \quad [0, 2\pi]$

**52.** $f(x) = \frac{x}{2} + \cos\frac{x}{2}, \quad [0, 4\pi]$

**53.** $f(x) = -3\sin\frac{x}{3}, \quad [0, 6\pi]$

**54.** $f(x) = 2\sin 3x + 4\cos 3x, \quad [0, \pi]$

### Comparing Functions

**In Exercises 55 and 56, use symmetry, extrema, and zeros to sketch the graph of $f$. How do the functions $f$ and $g$ differ?**

**55.** $f(x) = \frac{x^5 - 4x^3 + 3x}{x^2 - 1}$

$g(x) = x(x^2 - 3)$

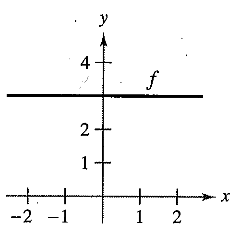
**56.** $f(t) = \cos^2 t - \sin^2 t$

$g(t) = 1 - 2\sin^2 t$

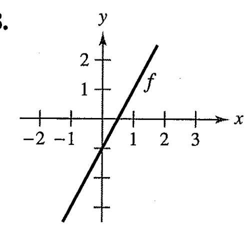
### Think About It

**In Exercises 57–62, the graph of $f$ is shown in the figure. Sketch a graph of the derivative of $f$. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

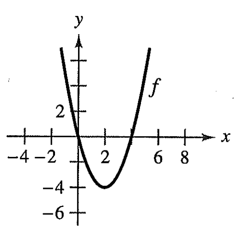
**57.**



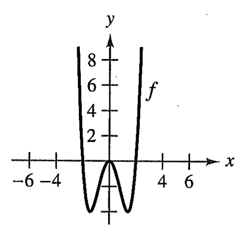

**58.**



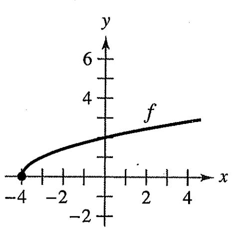

**59.**



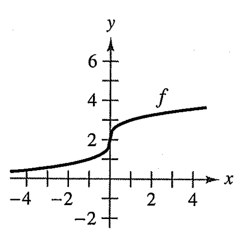

**60.**




**61.**




**62.**




### WRITING ABOUT CONCEPTS

**Transformations of Functions** **In Exercises 63–68, assume that $f$ is differentiable for all $x$. The signs of $f'$ are as follows.**

**$f'(x) > 0$ on $(-\infty, -4)$**

**$f'(x) < 0$ on $(-4, 6)$**

**$f'(x) > 0$ on $(6, \infty)$**

**Supply the appropriate inequality sign for the indicated value of $c$.**

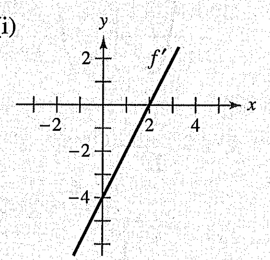
| | **Function** | **Sign of $g'(c)$** |
|---|---|---|
| **63.** | $g(x) = f(x) + 5$ | $g'(0) \; \square \; 0$ |
| **64.** | $g(x) = 3f(x) - 3$ | $g'(-5) \; \square \; 0$ |
| **65.** | $g(x) = -f(x)$ | $g'(-6) \; \square \; 0$ |
| **66.** | $g(x) = -f(x)$ | $g'(0) \; \square \; 0$ |
| **67.** | $g(x) = f(x - 10)$ | $g'(0) \; \square \; 0$ |
| **68.** | $g(x) = f(x - 10)$ | $g'(8) \; \square \; 0$ |

**69. Sketching a Graph** Sketch the graph of the arbitrary function $f$ such that

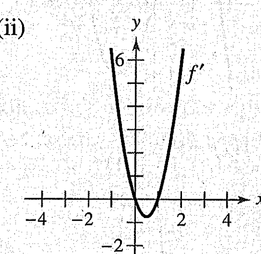
$$f'(x)\begin{cases} > 0, & x < 4 \\ \text{undefined}, & x = 4. \\ < 0, & x > 4 \end{cases}$$

**70. HOW DO YOU SEE IT?** Use the graph of $f'$ to (a) identify the critical numbers of $f$, (b) identify the open interval(s) on which $f$ is increasing or decreasing, and (c) determine whether $f$ has a relative maximum, a relative minimum, or neither at each critical number.

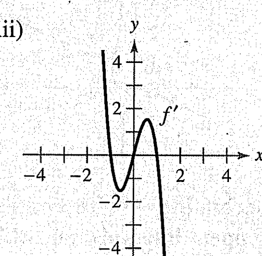
(i)



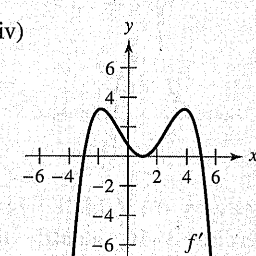

(ii)




(iii)




(iv)



71. **Analyzing a Critical Number** A differentiable function $f$ has one critical number at $x = 5$. Identify the relative extrema of $f$ at the critical number when $f'(4) = -2.5$ and $f'(6) = 3$.

72. **Analyzing a Critical Number** A differentiable function $f$ has one critical number at $x = 2$. Identify the relative extrema of $f$ at the critical number when $f'(1) = 2$ and $f'(3) = 6$.

**Think About It** **In Exercises 73 and 74, the function $f$ is differentiable on the indicated interval. The table shows $f'(x)$ for selected values of $x$. (a) Sketch the graph of $f$, (b) approximate the critical numbers, and (c) identify the relative extrema.**

73. $f$ is differentiable on $[-1, 1]$.

| $x$ | $-1$ | $-0.75$ | $-0.50$ | $-0.25$ | $0$ |
|---|---|---|---|---|---|
| $f'(x)$ | $-10$ | $-3.2$ | $-0.5$ | $0.8$ | $5.6$ |

| $x$ | $0.25$ | $0.50$ | $0.75$ | $1$ |
|---|---|---|---|---|
| $f'(x)$ | $3.6$ | $-0.2$ | $-6.7$ | $-20.1$ |

74. $f$ is differentiable on $[0, \pi]$.

| $x$ | $0$ | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ |
|---|---|---|---|---|---|
| $f'(x)$ | $3.14$ | $-0.23$ | $-2.45$ | $-3.11$ | $0.69$ |

| $x$ | $2\pi/3$ | $3\pi/4$ | $5\pi/6$ | $\pi$ |
|---|---|---|---|---|
| $f'(x)$ | $3.00$ | $1.37$ | $-1.14$ | $-2.84$ |

75. **Rolling a Ball Bearing** A ball bearing is placed on an inclined plane and begins to roll. The angle of elevation of the plane is $\theta$. The distance (in meters) the ball bearing rolls in $t$ seconds is $s(t) = 4.9(\sin\theta)t^2$.

(a) Determine the speed of the ball bearing after $t$ seconds.

(b) Complete the table and use it to determine the value of $\theta$ that produces the maximum speed at a particular time.

| $\theta$ | $0$ | $\pi/4$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $3\pi/4$ | $\pi$ |
|---|---|---|---|---|---|---|---|
| $s'(t)$ | | | | | | | |

76. **Modeling Data** The end-of-year assets of the Medicare Hospital Insurance Trust Fund (in billions of dollars) for the years 1999 through 2010 are shown.

1999: 141.4; 2000: 177.5; 2001: 208.7; 2002: 234.8;
2003: 256.0; 2004: 269.3; 2005: 285.8; 2006: 305.4
2007: 326.0; 2008: 321.3; 2009: 304.2; 2010: 271.9

*(Source: U.S. Centers for Medicare and Medicaid Services)*

(a) Use the regression capabilities of a graphing utility to find a model of the form $M = at^4 + bt^3 + ct^2 + dt + e$ for the data. (Let $t = 9$ represent 1999.)

(b) Use a graphing utility to plot the data and graph the model.

(c) Find the maximum value of the model and compare the result with the actual data.

77. **Numerical, Graphical, and Analytic Analysis** The concentration $C$ of a chemical in the bloodstream $t$ hours after injection into muscle tissue is

$$C(t) = \frac{3t}{27 + t^3}, \quad t \geq 0.$$

(a) Complete the table and use it to approximate the time when the concentration is greatest.

| $t$ | $0$ | $0.5$ | $1$ | $1.5$ | $2$ | $2.5$ | $3$ |
|---|---|---|---|---|---|---|---|
| $C(t)$ | | | | | | | |

(b) Use a graphing utility to graph the concentration function and use the graph to approximate the time when the concentration is greatest.

(c) Use calculus to determine analytically the time when the concentration is greatest.

78. **Numerical, Graphical, and Analytic Analysis** Consider the functions $f(x) = x$ and $g(x) = \sin x$ on the interval $(0, \pi)$.

(a) Complete the table and make a conjecture about which is the greater function on the interval $(0, \pi)$.

| $x$ | $0.5$ | $1$ | $1.5$ | $2$ | $2.5$ | $3$ |
|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | |
| $g(x)$ | | | | | | |

(b) Use a graphing utility to graph the functions and use the graphs to make a conjecture about which is the greater function on the interval $(0, \pi)$.

(c) Prove that $f(x) > g(x)$ on the interval $(0, \pi)$. [*Hint*: Show that $h'(x) > 0$, where $h = f - g$.]

79. **Trachea Contraction** Coughing forces the trachea (windpipe) to contract, which affects the velocity $v$ of the air passing through the trachea. The velocity of the air during coughing is

$$v = k(R - r)r^2, \quad 0 \leq r < R$$

where $k$ is a constant, $R$ is the normal radius of the trachea, and $r$ is the radius during coughing. What radius will produce the maximum air velocity?

80. **Electrical Resistance** The resistance $R$ of a certain type of resistor is

$$R = \sqrt{0.001T^4 - 4T + 100}$$

where $R$ is measured in ohms and the temperature $T$ is measured in degrees Celsius.

(a) Use a computer algebra system to find $dR/dT$ and the critical number of the function. Determine the minimum resistance for this type of resistor.

(b) Use a graphing utility to graph the function $R$ and use the graph to approximate the minimum resistance for this type of resistor.

**Motion Along a Line** **In Exercises 81–84, the function $s(t)$ describes the motion of a particle along a line. For each function, (a) find the velocity function of the particle at any time $t \geq 0$, (b) identify the time interval(s) in which the particle is moving in a positive direction, (c) identify the time interval(s) in which the particle is moving in a negative direction, and (d) identify the time(s) at which the particle changes direction.**

**81.** $s(t) = 6t - t^2$

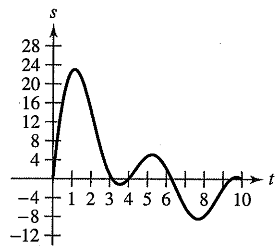
**82.** $s(t) = t^2 - 7t + 10$

**83.** $s(t) = t^3 - 5t^2 + 4t$

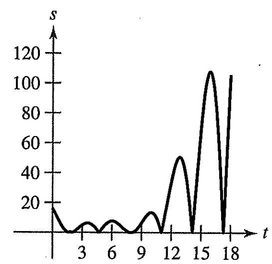
**84.** $s(t) = t^3 - 20t^2 + 128t - 280$

**Motion Along a Line** **In Exercises 85 and 86, the graph shows the position of a particle moving along a line. Describe how the particle's position changes with respect to time.**

**85.**




**86.**




**Creating Polynomial Functions** **In Exercises 87–90, find a polynomial function**

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

**that has only the specified extrema. (a) Determine the minimum degree of the function and give the criteria you used in determining the degree. (b) Using the fact that the coordinates of the extrema are solution points of the function, and that the $x$-coordinates are critical numbers, determine a system of linear equations whose solution yields the coefficients of the required function. (c) Use a graphing utility to solve the system of equations and determine the function. (d) Use a graphing utility to confirm your result graphically.**

**87.** Relative minimum: $(0, 0)$; Relative maximum: $(2, 2)$

**88.** Relative minimum: $(0, 0)$; Relative maximum: $(4, 1000)$

**89.** Relative minima: $(0, 0)$, $(4, 0)$; Relative maximum: $(2, 4)$

**90.** Relative minimum: $(1, 2)$; Relative maxima: $(-1, 4)$, $(3, 4)$

**True or False?** **In Exercises 91–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

**91.** The sum of two increasing functions is increasing.

**92.** The product of two increasing functions is increasing.

**93.** Every $n$th-degree polynomial has $(n - 1)$ critical numbers.

**94.** An $n$th-degree polynomial has at most $(n - 1)$ critical numbers.

**95.** There is a relative maximum or minimum at each critical number.

**96.** The relative maxima of the function $f$ are $f(1) = 4$ and $f(3) = 10$. Therefore, $f$ has at least one minimum for some $x$ in the interval $(1, 3)$.

**97. Proof** Prove the second case of Theorem 3.5.

**98. Proof** Prove the second case of Theorem 3.6.

**99. Proof** Use the definitions of increasing and decreasing functions to prove that $f(x) = x^3$ is increasing on $(-\infty, \infty)$.

**100. Proof** Use the definitions of increasing and decreasing functions to prove that

$$f(x) = \frac{1}{x}$$

is decreasing on $(0, \infty)$.

### PUTNAM EXAM CHALLENGE

**101.** Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers $x$.



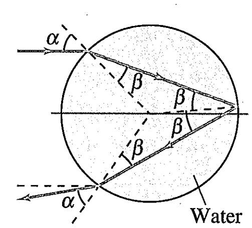
**SECTION PROJECT**

## Rainbows

Rainbows are formed when light strikes raindrops and is reflected and refracted, as shown in the figure. (This figure shows a cross section of a spherical raindrop.) The Law of Refraction states that

$$\frac{\sin \alpha}{\sin \beta} = k$$

where $k \approx 1.33$ (for water). The angle of deflection is given by $D = \pi + 2\alpha - 4\beta$.




(a) Use a graphing utility to graph

$$D = \pi + 2\alpha - 4\sin^{-1}\left(\frac{\sin \alpha}{k}\right), \quad 0 \leq \alpha \leq \frac{\pi}{2}.$$

(b) Prove that the minimum angle of deflection occurs when

$$\cos \alpha = \sqrt{\frac{k^2 - 1}{3}}.$$

For water, what is the minimum angle of deflection $D_{\min}$? (The angle $\pi - D_{\min}$ is called the *rainbow angle*.) What value of $\alpha$ produces this minimum angle? (A ray of sunlight that strikes a raindrop at this angle, $\alpha$, is called a *rainbow ray*.)

■ **FOR FURTHER INFORMATION** For more information about the mathematics of rainbows, see the article "Somewhere Within the Rainbow" by Steven Janke in *The UMAP Journal*.