3.4 Exercise Problems Log Derivatives and Log Differentiation

p.259-261 #17-75 odd

Find the derivative.

Log Properties:

i) $\ln a + \ln b = \ln(ab)$

ii) $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

iii) $n \cdot \ln a = \ln a^n$

Derivative Rules:

i) $\frac{d}{dx}\ln u = \frac{u'}{u}$

ii) $\frac{d}{dx}e^u = e^u \cdot u'$

iii) $\frac{d}{dx}\log_a u = \frac{1}{\ln a}\cdot\frac{u'}{u}$

17) $f(x) = \overbrace{x}^{f}\,\overbrace{\ln(x^2+4)}^{g}$ *product Rule

$$f'(x) = \overbrace{(1)}^{f'}\overbrace{\ln(x^2+4)}^{g} + \overbrace{x}^{f}\cdot\overbrace{\frac{2x}{x^2+4}}^{g'}$$

$$\boxed{f'(x) = \ln(x^2+4) + \frac{2x^2}{x^2+4}}$$

19) $f(x) = \overbrace{x^2}^{f}\,\overbrace{\log_5(3x+5)}^{g}$

$$f'(x) = \overbrace{2x}^{f'}\cdot\overbrace{\log_5(3x+5)}^{g} + \overbrace{x^2}^{f}\cdot\overbrace{\frac{1}{\ln 5}\cdot\frac{3}{3x+5}}^{g'}$$

$$\boxed{f'(x) = 2x\log_5(3x+5) + \frac{3x^2}{\ln 5(3x+5)}}$$

21) $f(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ ← expand log expression before applying derivative $\left(\ln\frac{a}{b} = \ln a - \ln b\right)$

$$f(x) = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$$

$$f'(x) = \frac{1}{2}\cdot\frac{1}{1+x} - \frac{1}{2}\cdot\frac{-1}{1-x}$$

$$\boxed{f'(x) = \frac{1}{2(1+x)} + \frac{1}{2(1-x)}}$$

23) $f(x) = \ln(\ln x)$ *chain Rule out: $\ln(\ )$ in: $\ln x$

$$f'(x) = \frac{\frac{1}{x}}{\ln x} \rightarrow \boxed{f'(x) = \frac{1}{x\ln x}}$$

25) $g(x) = \ln\left(\frac{x}{\sqrt{x^2+1}}\right)$ ↙ expand log expression first

$g(x) = \ln\left(\frac{x}{(x^2+1)^{1/2}}\right) = \ln x - \ln(x^2+1)^{1/2}$

$g(x) = \ln x - \frac{1}{2}\ln(x^2+1)$

$g'(x) = \frac{1}{x} - \frac{1}{2}\cdot\frac{2x}{x^2+1} \rightarrow \boxed{g'(x) = \frac{1}{x} - \frac{x}{x^2+1}}$

27) $f(x) = \ln\left(\frac{(x^2+1)^2}{x\sqrt{x^2-1}}\right)$ ↙ expand first

$f(x) = \ln(x^2+1)^2 - \ln x - \ln\sqrt{x^2-1} = \ln(x^2+1)^2 - \ln x - \ln(x^2-1)^{1/2}$

$f(x) = 2\ln(x^2+1) - \ln x - \frac{1}{2}\ln(x^2-1)$

$f'(x) = 2\left(\frac{2x}{x^2+1}\right) - \frac{1}{x} - \frac{1}{2}\left(\frac{2x}{x^2-1}\right) \rightarrow \boxed{f'(x) = \frac{4x}{x^2+1} - \frac{1}{x} - \frac{x}{x^2-1}}$

29) $F(\theta) = \ln(\sin\theta)$ * chain Rule out: $\ln(\ )$ in: $\sin\theta$

$F'(\theta) = \frac{\cos\theta}{\sin\theta} \rightarrow \boxed{F'(\theta) = \cot\theta}$

31) $g(x) = \ln(x + \sqrt{x^2+4})$ *cannot expand! * chain Rule ↗ out: $\ln(\ )$ in: $x + (x^2+4)^{1/2}$

$g'(x) = \frac{1 + \frac{1}{2}(x^2+4)^{-1/2}(2x)}{x + (x^2+4)^{1/2}} \rightarrow \frac{1 + \frac{x}{(x^2+4)^{1/2}}}{x + (x^2+4)^{1/2}} \rightarrow \frac{\frac{(x^2+4)^{1/2}}{(x^2+4)^{1/2}} + \frac{x}{(x^2+4)^{1/2}}}{x + (x^2+4)^{1/2}}$

$g'(x) = \frac{\frac{(x^2+4)^{1/2}+x}{(x^2+4)^{1/2}}}{x + (x^2+4)^{1/2}} \rightarrow \frac{\cancel{(x^2+4)^{1/2}+x}}{(x^2+4)^{1/2}} \cdot \frac{1}{\cancel{x+(x^2+4)^{1/2}}} = \boxed{\frac{1}{\sqrt{x^2+4}}}$

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33) $f(x) = \log_2(1+x^2)$ $\quad$ * $\frac{d}{dx}\log_a u = \frac{1}{\ln a}\cdot\frac{u'}{u}$

$$f'(x) = \frac{1}{\ln 2}\cdot\frac{2x}{1+x^2} \rightarrow \boxed{f'(x) = \frac{2x}{\ln 2(1+x^2)}}$$

35) $f(x) = \tan^{-1}(\ln x)$ $\quad$ *chain Rule $\quad$ out: $\tan^{-1}(\;)$ $\quad$ in: $\ln x$ $\quad$ * $\frac{d}{dx}\tan^{-1}(u) = \frac{u'}{1+u^2}$

$$f'(x) = \frac{\frac{1}{x}}{1+(\ln x)^2} \rightarrow \boxed{f'(x) = \frac{1}{x[1+(\ln x)^2]}}$$

37) $s(t) = \ln(\tan^{-1}(t))$ $\quad$ *chain Rule $\quad$ out: $\ln(\;)$ $\quad$ in: $\tan^{-1}(t)$

$$s'(t) = \frac{\frac{1}{1+t^2}}{\tan^{-1}(t)} \rightarrow \boxed{s'(t) = \frac{1}{\tan^{-1}(t)(1+t^2)}}$$

39) $G(x) = (\ln x)^{1/2}$ $\quad$ *cannot expand $\quad$ *chain Rule $\rightarrow$ out: $(\;)^{1/2}$ $\quad$ in: $\ln x$

$$G'(x) = \frac{1}{2}(\quad)^{-1/2}\cdot\left(\frac{1}{x}\right)$$

$$G'(x) = \frac{1}{2}(\ln x)^{-1/2}\cdot\left(\frac{1}{x}\right)$$

$$\boxed{G'(x) = \frac{1}{2x\sqrt{\ln x}}} \quad \text{or} \quad \frac{1}{2x(\ln x)^{1/2}}$$

41) $f(\theta) = \sin(\ln\theta)$ $\quad$ *chain Rule $\quad$ out: $\sin(\;)$ $\quad$ in: $\ln\theta$

$$f'(\theta) = \cos(\ln\theta)\cdot\frac{1}{\theta}$$

$$\boxed{f'(\theta) = \frac{\cos(\ln\theta)}{\theta}}$$

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$43)\ g(x) = (\log_3 x)^{3/2}$

* cannot expand
* chain Rule → out: $(\ )^{3/2}$ in: $\log_3 x$

$* \frac{d}{dx}\log_a u = \frac{1}{\ln a}\cdot\frac{u'}{u}$

$$g'(x) = \frac{3}{2}(\quad)^{1/2}\cdot\frac{1}{\ln 3}\cdot\frac{1}{x}$$

$$g'(x) = \frac{3}{2}(\log_3 x)^{1/2}\cdot\frac{1}{x\ln 3} \rightarrow \boxed{g'(x) = \frac{3(\log_3 x)^{1/2}}{2x\ln 3}}$$

$\boxed{\text{Find } \frac{dy}{dx}}$

$$45)\ \underbrace{x}_{f}\underbrace{\ln y}_{g} + \underbrace{y}_{f}\underbrace{\ln x}_{g} = 2$$

* implicit
* product Rule

$$\underbrace{(1)}_{f'}\underbrace{(\ln y)}_{g} + \underbrace{(x)}_{f}\cdot\underbrace{\frac{1}{y}\left(\frac{dy}{dx}\right)}_{g'} + \underbrace{\left(\frac{dy}{dx}\right)}_{f'}\underbrace{\ln x}_{g} + \underbrace{y}_{f}\cdot\underbrace{\left(\frac{1}{x}\right)}_{g'} = 0$$

$$\frac{x}{y}\left(\frac{dy}{dx}\right) + \ln y + \ln x\left(\frac{dy}{dx}\right) + \frac{y}{x} = 0$$

$$\frac{dy}{dx}\left[\frac{x}{y} + \ln x\right] = -\frac{y}{x} - \ln y$$

$$\frac{dy}{dx} = \frac{-\frac{y}{x} - \ln y}{\frac{x}{y} + \ln x} \rightarrow \frac{-\frac{y}{x} - \frac{x\ln y}{x}}{\frac{x}{y} + \frac{y\ln x}{y}}$$

$$\frac{dy}{dx} = \frac{\frac{-y - x\ln y}{x}}{\frac{x + y\ln x}{y}} \rightarrow \frac{-y - x\ln y}{x}\cdot\frac{y}{x + y\ln x}$$

$$\boxed{\frac{dy}{dx} = \frac{-y^2 - xy\ln y}{x^2 + xy\ln x}}$$

3.4 Find $\frac{dy}{dx}$

* chain Rule
* implicit

out: ln( )
in: x^2+y^2

47) $\ln(x^2+y^2) = x+y$

$$\frac{2x+2y\left(\frac{dy}{dx}\right)}{x^2+y^2} = \frac{1+\frac{dy}{dx}}{1}$$

$$2x+2y\left(\frac{dy}{dx}\right) = (x^2+y^2)\left(1+\frac{dy}{dx}\right)$$

$$2x+2y\left(\frac{dy}{dx}\right) = x^2+x^2\left(\frac{dy}{dx}\right)+y^2+y^2\left(\frac{dy}{dx}\right)$$

$$2y\left(\frac{dy}{dx}\right)-x^2\left(\frac{dy}{dx}\right)-y^2\left(\frac{dy}{dx}\right) = x^2+y^2-2x$$

$$\frac{dy}{dx}(2y-x^2-y^2) = x^2+y^2-2x$$

$$\boxed{\frac{dy}{dx} = \frac{x^2+y^2-2x}{2y-x^2-y^2}}$$

49) $\ln\left(\frac{y}{x}\right) = y$

* expand first
* implicit diff.

$$\ln y - \ln x = y$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) - \frac{1}{x} = 1\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}\left(\frac{1}{y}\right) - 1\left(\frac{dy}{dx}\right) = \frac{1}{x}$$

$$\frac{dy}{dx}\left(\frac{1}{y}-1\right) = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x}}{\frac{1}{y}-1} \rightarrow \frac{\frac{1}{x}}{\frac{1}{y}-\frac{y}{y}}$$

$$= \frac{\frac{1}{x}}{\frac{1-y}{y}} \rightarrow \frac{1}{x}\cdot\frac{y}{1-y}$$

$$\boxed{\frac{dy}{dx} = \frac{y}{x(1-y)}}$$

$\boxed{3.4}$ # 51-69 use $\boxed{\text{Log Differentiation}}$

(Introduce Logs into an equation with NO Logs to begin with)

51) $y = (x^2+1)^2(2x^3-1)^4$

$\ln y = \ln\left[(x^2+1)^2(2x^3-1)^4\right]$

$\ln y = \ln(x^2+1)^2 + \ln(2x^3-1)^4$

$\ln y = 2\ln(x^2+1) + 4\ln(2x^3-1)$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = 2\cdot\frac{2x}{x^2+1} + 4\cdot\frac{6x^2}{2x^3-1}$

$\frac{dy}{dx} = y\left[\frac{4x}{x^2+1} + \frac{24x^2}{2x^3-1}\right]$

$\boxed{\frac{dy}{dx} = (x^2+1)^2(2x^3-1)^4\left(\frac{4x}{x^2+1} + \frac{24x^2}{2x^3-1}\right)}$

53) $y = \frac{x^2(x^3+1)}{\sqrt{x^2+1}}$

$\ln y = \ln\left[\frac{x^2(x^3+1)}{(x^2+1)^{1/2}}\right]$

$\ln y = \ln x^2 + \ln(x^3+1) - \ln(x^2+1)^{1/2}$

$\ln y = 2\ln x + \ln(x^3+1) - \frac{1}{2}\ln(x^2+1)$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = 2\left(\frac{1}{x}\right) + \frac{3x^2}{x^3+1} - \frac{1}{2}\cdot\frac{2x}{x^2+1}$

$\frac{dy}{dx} = y\left[\frac{2}{x} + \frac{3x^2}{x^3+1} - \frac{x}{x^2+1}\right]$

$\boxed{\frac{dy}{dx} = \frac{x^2(x^3+1)}{\sqrt{x^2+1}}\left[\frac{2}{x} + \frac{3x^2}{x^3+1} - \frac{x}{x^2+1}\right]}$

55) $y = \frac{x\cos x}{(x^2+1)^3\sin x}$

$\ln y = \ln\left[\frac{x\cos x}{(x^2+1)^3\sin x}\right]$

$\ln y = \ln x + \ln(\cos x) - \ln(x^2+1)^3 - \ln(\sin x)$

$\ln y = \ln x + \ln(\cos x) - 3\ln(x^2+1) - \ln(\sin x)$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{-\sin x}{\cos x} - 3\left(\frac{2x}{x^2+1}\right) - \frac{\cos x}{\sin x}$

$\frac{dy}{dx} = y\left[\frac{1}{x} - \tan x - \frac{6x}{x^2+1} - \cot x\right]$

$\boxed{\frac{dy}{dx} = \frac{x\cos x}{(x^2+1)^3\sin x}\left[\frac{1}{x} - \tan x - \frac{6x}{x^2+1} - \cot x\right]}$

3.4 Log Differentiation (continued)

Find dy/dx

*Use Log differentiation whenever the expression is in the form of

$$\boxed{y=(\text{variable})^{\text{variable}} \quad \text{ex: } y=x^x}$$

57) $y=(3x)^x$

$\ln y=\ln(3x)^x$

$\ln y = x\ln(3x)$

$\ln y = \underbrace{x}_{f}\cdot\underbrace{\ln(3x)}_{g}$ *product Rule

$\frac{1}{y}\left(\frac{dy}{dx}\right)=\underbrace{(1)}_{f'}\cdot\underbrace{\ln(3x)}_{g}+\underbrace{x}_{f}\cdot\underbrace{\frac{3}{3x}}_{g'}$

$\frac{dy}{dx}=y\left[\ln(3x)+\frac{3x}{3x}\right]$

$$\boxed{\frac{dy}{dx}=(3x)^x\left[\ln(3x)+1\right]}$$

59) $y=x^{\ln x}$

$\ln y=\ln x^{(\ln x)}$

$\ln y=(\ln x)(\ln x)$ ← product Rule

or

$\ln y=(\ln x)^2$ ← chain Rule out: $(\ )^2$ in: $\ln x$

$\frac{1}{y}\left(\frac{dy}{dx}\right)=2(\ln x)\cdot\frac{1}{x}$

$\frac{dy}{dx}=y\cdot 2(\ln x)\cdot\frac{1}{x}$

$\frac{dy}{dx}=x^{\ln x}\cdot 2\ln x\cdot\frac{1}{x}$

$$\boxed{\frac{dy}{dx}=\frac{2x^{\ln x}(\ln x)}{x}}$$

61) $y=x^{x^2}$

$\ln y=\ln x^{x^2}$

$\ln y = x^2\cdot\ln x$

$\ln y=\underbrace{x^2}_{f}\cdot\underbrace{\ln x}_{g}$

$\frac{1}{y}\left(\frac{dy}{dx}\right)=\underbrace{2x}_{f'}\cdot\underbrace{\ln x}_{g}+\underbrace{x^2}_{f}\cdot\underbrace{\frac{1}{x}}_{g'}$

$\frac{dy}{dx}=y\left[2x\ln x+x\right]$

$$\boxed{\frac{dy}{dx}=x^{x^2}\left(2x\ln x+x\right)}$$

$\boxed{3.4}$

63) $y = x^{e^x}$

$\ln y = \ln x^{e^x}$

$\ln y = e^x \ln x$

$\ln y = \overbrace{e^x}^{f}\overbrace{\ln x}^{g}$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = \overbrace{e^x}^{f'} \cdot \overbrace{\ln x}^{g} + \overbrace{e^x}^{f} \cdot \overbrace{\frac{1}{x}}^{g'}$

$\frac{dy}{dx} = y\left[e^x \ln x + \frac{e^x}{x}\right]$

$$\boxed{\frac{dy}{dx} = x^{e^x}\left[e^x \ln x + \frac{e^x}{x}\right]}$$

65) $y = x^{\sin x}$

$\ln y = \ln x^{\sin x}$

$\ln y = \overbrace{(\sin x)}^{f}\overbrace{(\ln x)}^{g}$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = \overbrace{\cos x}^{f'} \cdot \overbrace{\ln x}^{g} + \overbrace{\sin x}^{f} \cdot \overbrace{\frac{1}{x}}^{g'}$

$\frac{dy}{dx} = y\left(\cos x(\ln x) + \frac{\sin x}{x}\right)$

$$\boxed{\frac{dy}{dx} = x^{\sin x}\left[\cos x(\ln x) + \frac{\sin x}{x}\right]}$$

67) $y = (\sin x)^x$

$\ln y = \ln(\sin x)^x$

$\ln y = \overbrace{x}^{f} \cdot \overbrace{\ln(\sin x)}^{g}$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = \overbrace{(1)}^{f'} \overbrace{\ln(\sin x)}^{g} + \overbrace{x}^{f} \cdot \overbrace{\frac{\cos x}{\sin x}}^{g'}$

$\frac{dy}{dx} = y\left[\ln(\sin x) + \frac{x\cos x}{\sin x}\right]$

$$\boxed{\frac{dy}{dx} = (\sin x)^x\left[\ln(\sin x) + x\cot x\right]}$$

69) $y = (\sin x)^{\cos x}$

$\ln y = \ln(\sin x)^{\cos x}$

$\ln y = \overbrace{\cos x}^{f} \cdot \overbrace{\ln(\sin x)}^{g}$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = \overbrace{-\sin x}^{f'} \cdot \overbrace{\ln(\sin x)}^{g} + \overbrace{\cos x}^{f} \cdot \overbrace{\frac{\cos x}{\sin x}}^{g'}$

$\frac{dy}{dx} = y\left[-\sin x \ln(\sin x) + \cos x \cot x\right]$

$$\boxed{\frac{dy}{dx} = (\sin x)^{\cos x}\left[-\sin x \cdot \ln(\sin x) + \cos x \cot x\right]}$$

$\boxed{3.4}$ Find $\frac{dy}{dx}$

71) $x^y = 4$

$\ln x^y = \ln 4$

$\overbrace{y}^{f}\overbrace{\ln x}^{g} = \ln 4$

$\boxed{*\, y = \frac{\ln 4}{\ln x}}$ substitution for final step

$\underbrace{\left(\frac{dy}{dx}\right)}_{f'}\underbrace{\ln x}_{g} + \underbrace{y}_{f}\underbrace{\left(\frac{1}{x}\right)}_{g'} = 0$

$\frac{dy}{dx}\ln x = \frac{-y}{x}$

$\frac{dy}{dx} = \frac{-y}{x}\cdot\frac{1}{\ln x}$

$*\, y = \frac{\ln 4}{\ln x}$

$\frac{dy}{dx} = \frac{-\frac{\ln 4}{\ln x}}{x}\cdot\frac{1}{\ln x}$

$\frac{dy}{dx} = \frac{-\ln 4}{x\ln 4}\cdot\frac{1}{\ln 4}$

$\boxed{\frac{dy}{dx} = \frac{-\ln 4}{x(\ln 4)^2}}$

Find equation of tangent line

73) $y = \ln(5x)$ at $\left(\frac{1}{5}, 0\right)$

$y' = \frac{5}{5x} \rightarrow y' = \frac{1}{x} \rightarrow y'\left(\frac{1}{5}\right) = \frac{1}{1/5} = 5$

point: $(1/5, 0)$

slope: $m = 5$

$\boxed{y - 0 = 5(x - 1/5)}$

*use log differentiation for this equation

75) $y = \frac{x^2\sqrt{3x-2}}{(x-1)^2}$ at $(2, 8)$

$\ln y = \ln\left[\frac{x^2(3x-2)^{1/2}}{(x-1)^2}\right]$

$\ln y = \ln x^2 + \ln(3x-2)^{1/2} - \ln(x-1)^2$

$\ln y = 2\ln x + \frac{1}{2}\ln(3x-2) - 2\ln(x-1)$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = 2\left(\frac{1}{x}\right) + \frac{1}{2}\left(\frac{3}{3x-2}\right) - 2\left(\frac{1}{x-1}\right)$

$\frac{dy}{dx} = y\left[\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1}\right]$

$\frac{dy}{dx} = \frac{x^2\sqrt{3x-2}}{(x-1)^2}\left[\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1}\right]$

$\left.\frac{dy}{dx}\right|_{(2,8)} = \frac{2^2\sqrt{6-2}}{(2-1)^2}\left[\frac{2}{2} + \frac{3}{2(6-2)} - \frac{2}{2-1}\right]$

$= 4\sqrt{4}\left(1 + \frac{3}{8} - 2\right) = \boxed{-5}$

point: $(2, 8)$

slope: $m = -5$

$\boxed{y - 8 = -5(x-2)}$