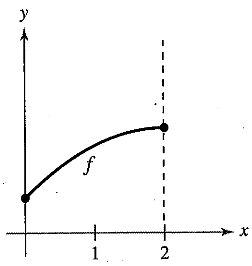
3.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Using a Graph In Exercises 1 and 2, the graph of f is shown. State the signs of f' and f'' on the interval $(0, 2)$.

1.


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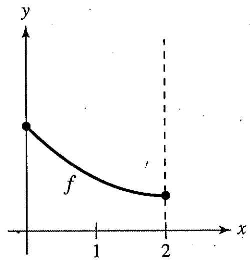


Determining Concavity In Exercises 3–14, determine the open intervals on which the graph is concave upward or concave downward.

3. $y = x^2 - x - 2$
4. $g(x) = 3x^2 - x^3$
5. $f(x) = -x^3 + 6x^2 - 9x - 1$
6. $h(x) = x^5 - 5x + 2$
7. $f(x) = \dfrac{24}{x^2 + 12}$
8. $f(x) = \dfrac{2x^2}{3x^2 + 1}$
9. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$
10. $y = \dfrac{-3x^5 + 40x^3 + 135x}{270}$
11. $g(x) = \dfrac{x^2 + 4}{4 - x^2}$
12. $h(x) = \dfrac{x^2 - 1}{2x - 1}$
13. $y = 2x - \tan x, \quad \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
14. $y = x + \dfrac{2}{\sin x}, \quad (-\pi, \pi)$

Finding Points of Inflection In Exercises 15–30, find the points of inflection and discuss the concavity of the graph of the function.

15. $f(x) = x^3 - 6x^2 + 12x$
16. $f(x) = -x^3 + 6x^2 - 5$
17. $f(x) = \frac{1}{2}x^4 + 2x^3$
18. $f(x) = 4 - x - 3x^4$
19. $f(x) = x(x - 4)^3$
20. $f(x) = (x - 2)^3(x - 1)$
21. $f(x) = x\sqrt{x + 3}$
22. $f(x) = x\sqrt{9 - x}$
23. $f(x) = \dfrac{4}{x^2 + 1}$
24. $f(x) = \dfrac{x + 3}{\sqrt{x}}$
25. $f(x) = \sin \dfrac{x}{2}, \quad [0, 4\pi]$
26. $f(x) = 2 \csc \dfrac{3x}{2}, \quad (0, 2\pi)$
27. $f(x) = \sec\left(x - \dfrac{\pi}{2}\right), \quad (0, 4\pi)$
28. $f(x) = \sin x + \cos x, \quad [0, 2\pi]$
29. $f(x) = 2 \sin x + \sin 2x, \quad [0, 2\pi]$
30. $f(x) = x + 2 \cos x, \quad [0, 2\pi]$

Using the Second Derivative Test In Exercises 31–42, find all relative extrema. Use the Second Derivative Test where applicable.

31. $f(x) = 6x - x^2$
32. $f(x) = x^2 + 3x - 8$
33. $f(x) = x^3 - 3x^2 + 3$
34. $f(x) = -x^3 + 7x^2 - 15x$
35. $f(x) = x^4 - 4x^3 + 2$
36. $f(x) = -x^4 + 4x^3 + 8x^2$
37. $f(x) = x^{2/3} - 3$
38. $f(x) = \sqrt{x^2 + 1}$
39. $f(x) = x + \dfrac{4}{x}$
40. $f(x) = \dfrac{x}{x - 1}$
41. $f(x) = \cos x - x, \quad [0, 4\pi]$
42. $f(x) = 2 \sin x + \cos 2x, \quad [0, 2\pi]$

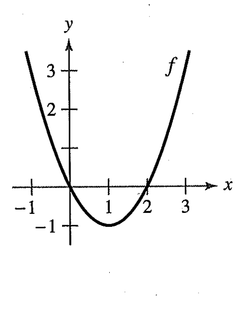
Finding Extrema and Points of Inflection Using Technology In Exercises 43–46, use a computer algebra system to analyze the function over the given interval. (a) Find the first and second derivatives of the function. (b) Find any relative extrema and points of inflection. (c) Graph f, f', and f'' on the same set of coordinate axes and state the relationship between the behavior of f and the signs of f' and f''.

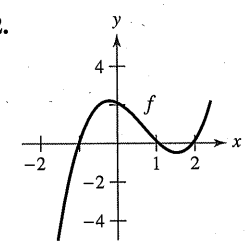
43. $f(x) = 0.2x^2(x - 3)^3, \quad [-1, 4]$
44. $f(x) = x^2\sqrt{6 - x^2}, \quad [-\sqrt{6}, \sqrt{6}]$
45. $f(x) = \sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x, \quad [0, \pi]$
46. $f(x) = \sqrt{2x} \sin x, \quad [0, 2\pi]$

WRITING ABOUT CONCEPTS

47. **Sketching a Graph** Consider a function f such that f' is increasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.
48. **Sketching a Graph** Consider a function f such that f' is decreasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.
49. **Sketching a Graph** Sketch the graph of a function f that does *not* have a point of inflection at $(c, f(c))$ even though $f''(c) = 0$.
50. **Think About It** S represents weekly sales of a product. What can be said of S' and S'' for each of the following statements?
(a) The rate of change of sales is increasing.
(b) Sales are increasing at a slower rate.
(c) The rate of change of sales is constant.
(d) Sales are steady.
(e) Sales are declining, but at a slower rate.
(f) Sales have bottomed out and have started to rise.

Sketching Graphs In Exercises 51 and 52, the graph of f is shown. Graph f, f', and f'' on the same set of coordinate axes. To print an enlarged copy of the graph, go to *MathGraphs.com*.

51.


52.


Think About It In Exercises 53–56, sketch the graph of a function f having the given characteristics.

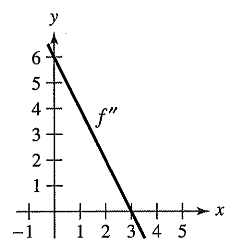
53. $f(2) = f(4) = 0$
$f'(x) < 0$ for $x < 3$
$f'(3)$ does not exist.
$f'(x) > 0$ for $x > 3$
$f''(x) < 0, x \neq 3$

54. $f(0) = f(2) = 0$
$f'(x) > 0$ for $x < 1$
$f'(1) = 0$
$f'(x) < 0$ for $x > 1$
$f''(x) < 0$

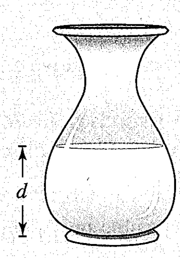
55. $f(2) = f(4) = 0$
$f'(x) > 0$ for $x < 3$
$f'(3)$ does not exist.
$f'(x) < 0$ for $x > 3$
$f''(x) > 0, x \neq 3$

56. $f(0) = f(2) = 0$
$f'(x) < 0$ for $x < 1$
$f'(1) = 0$
$f'(x) > 0$ for $x > 1$
$f''(x) > 0$

57. **Think About It** The figure shows the graph of f''. Sketch a graph of f. (The answer is not unique.) To print an enlarged copy of the graph, go to *MathGraphs.com*.



58. **HOW DO YOU SEE IT?** Water is running into the vase shown in the figure at a constant rate.



(a) Graph the depth d of water in the vase as a function of time.
(b) Does the function have any extrema? Explain.
(c) Interpret the inflection points of the graph of d.

59. **Conjecture** Consider the function

$$f(x) = (x - 2)^n.$$

(a) Use a graphing utility to graph f for $n = 1, 2, 3$, and 4. Use the graphs to make a conjecture about the relationship between n and any inflection points of the graph of f.
(b) Verify your conjecture in part (a).

60. **Inflection Point** Consider the function $f(x) = \sqrt[3]{x}$.

(a) Graph the function and identify the inflection point.
(b) Does $f''(x)$ exist at the inflection point? Explain.

Finding a Cubic Function In Exercises 61 and 62, find a, b, c, and d such that the cubic

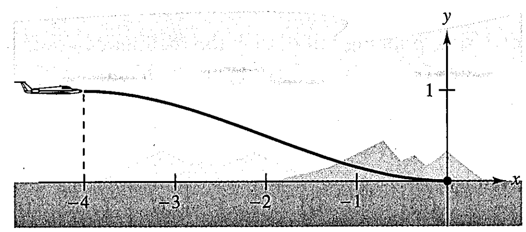
$$f(x) = ax^3 + bx^2 + cx + d$$

satisfies the given conditions.

61. Relative maximum: $(3, 3)$
Relative minimum: $(5, 1)$
Inflection point: $(4, 2)$

62. Relative maximum: $(2, 4)$
Relative minimum: $(4, 2)$
Inflection point: $(3, 3)$

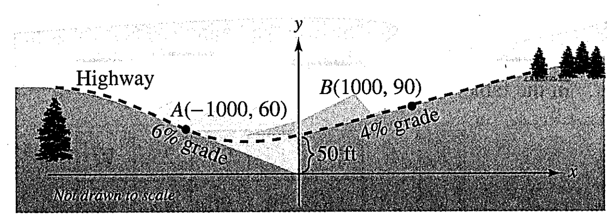
63. **Aircraft Glide Path** A small aircraft starts its descent from an altitude of 1 mile, 4 miles west of the runway (see figure).



(a) Find the cubic $f(x) = ax^3 + bx^2 + cx + d$ on the interval $[-4, 0]$ that describes a smooth glide path for the landing.
(b) The function in part (a) models the glide path of the plane. When would the plane be descending at the greatest rate?

■ **FOR FURTHER INFORMATION** For more information on this type of modeling, see the article "How Not to Land at Lake Tahoe!" by Richard Barshinger in *The American Mathematical Monthly*. To view this article, go to *MathArticles.com*.

64. **Highway Design** A section of highway connecting two hillsides with grades of 6% and 4% is to be built between two points that are separated by a horizontal distance of 2000 feet (see figure). At the point where the two hillsides come together, there is a 50-foot difference in elevation.



(a) Design a section of highway connecting the hillsides modeled by the function

$$f(x) = ax^3 + bx^2 + cx + d, \quad -1000 \leq x \leq 1000.$$

At points A and B, the slope of the model must match the grade of the hillside.

(b) Use a graphing utility to graph the model.
(c) Use a graphing utility to graph the derivative of the model.
(d) Determine the grade at the steepest part of the transitional section of the highway.

65. Average Cost A manufacturer has determined that the total cost C of operating a factory is

$$C = 0.5x^2 + 15x + 5000$$

where x is the number of units produced. At what level of production will the average cost per unit be minimized? (The average cost per unit is C/x.)

66. Specific Gravity A model for the specific gravity of water S is

$$S = \frac{5.755}{10^8}T^3 - \frac{8.521}{10^6}T^2 + \frac{6.540}{10^5}T + 0.99987, \ 0 < T < 25$$

where T is the water temperature in degrees Celsius.

(a) Use the second derivative to determine the concavity of S.

(b) Use a computer algebra system to find the coordinates of the maximum value of the function.

(c) Use a graphing utility to graph the function over the specified domain. (Use a setting in which $0.996 \le S \le 1.001$.)

(d) Estimate the specific gravity of water when $T = 20°$.

67. Sales Growth The annual sales S of a new product are given by

$$S = \frac{5000t^2}{8 + t^2}, \quad 0 \le t \le 3$$

where t is time in years.

(a) Complete the table. Then use it to estimate when the annual sales are increasing at the greatest rate.

t	0.5	1	1.5	2	2.5	3
S						

(b) Use a graphing utility to graph the function S. Then use the graph to estimate when the annual sales are increasing at the greatest rate.

(c) Find the exact time when the annual sales are increasing at the greatest rate.

68. Modeling Data The average typing speed S (in words per minute) of a typing student after t weeks of lessons is shown in the table.

t	5	10	15	20	25	30
S	38	56	79	90	93	94

A model for the data is

$$S = \frac{100t^2}{65 + t^2}, \quad t > 0.$$

(a) Use a graphing utility to plot the data and graph the model.

(b) Use the second derivative to determine the concavity of S. Compare the result with the graph in part (a).

(c) What is the sign of the first derivative for $t > 0$? By combining this information with the concavity of the model, what inferences can be made about the typing speed as t increases?

Linear and Quadratic Approximations In Exercises 69–72, use a graphing utility to graph the function. Then graph the linear and quadratic approximations

$$P_1(x) = f(a) + f'(a)(x - a)$$

and

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2$$

in the same viewing window. Compare the values of f, P_1, and P_2 and their first derivatives at $x = a$. How do the approximations change as you move farther away from $x = a$?

Function	Value of a
69. $f(x) = 2(\sin x + \cos x)$	$a = \frac{\pi}{4}$
70. $f(x) = 2(\sin x + \cos x)$	$a = 0$
71. $f(x) = \sqrt{1 - x}$	$a = 0$
72. $f(x) = \frac{\sqrt{x}}{x - 1}$	$a = 2$

73. Determining Concavity Use a graphing utility to graph

$$y = x \sin \frac{1}{x}.$$

Show that the graph is concave downward to the right of

$$x = \frac{1}{\pi}.$$

74. Point of Inflection and Extrema Show that the point of inflection of

$$f(x) = x(x - 6)^2$$

lies midway between the relative extrema of f.

True or False? In Exercises 75–78, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

75. The graph of every cubic polynomial has precisely one point of inflection.

76. The graph of

$$f(x) = \frac{1}{x}$$

is concave downward for $x < 0$ and concave upward for $x > 0$, and thus it has a point of inflection at $x = 0$.

77. If $f'(c) > 0$, then f is concave upward at $x = c$.

78. If $f''(2) = 0$, then the graph of f must have a point of inflection at $x = 2$.

Proof In Exercises 79 and 80, let f and g represent differentiable functions such that $f'' \neq 0$ and $g'' \neq 0$.

79. Show that if f and g are concave upward on the interval (a, b), then $f + g$ is also concave upward on (a, b).

80. Prove that if f and g are positive, increasing, and concave upward on the interval (a, b), then fg is also concave upward on (a, b).