39. $f(x) = x + \dfrac{4}{x}$

$f'(x) = 1 - \dfrac{4}{x^2} = \dfrac{x^2 - 4}{x^2}$

$f''(x) = \dfrac{8}{x^3}$

Critical numbers: $x = \pm 2$

$f''(-2) = -1 < 0$

Therefore, $(-2, -4)$ is a relative maximum.

$f''(2) = 1 > 0$

Therefore, $(2, 4)$ is a relative minimum.

40. $f(x) = \dfrac{x}{x - 1}$

$f'(x) = \dfrac{-1}{(x - 1)^2}$

There are no critical numbers and $x = 1$ is not in the domain. There are no relative extrema.

41. $f(x) = \cos x - x, 0 \le x \le 4\pi$

$f'(x) = -\sin x - 1 \le 0$

Therefore, f is non-increasing and there are no relative extrema.

42. $f(x) = 2\sin x + \cos 2x, 0 \le x \le 2\pi$

$f'(x) = 2\cos x - 2\sin 2x = 2\cos x - 4\sin x \cos x$

$= 2\cos x(1 - 2\sin x) = 0$ when $x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$.

$f''(x) = -2\sin x - 4\cos 2x$

$f''\left(\dfrac{\pi}{6}\right) = -3 < 0$

Therefore, $\left(\dfrac{\pi}{6}, \dfrac{3}{2}\right)$ is a relative maximum.

$f''\left(\dfrac{\pi}{2}\right) = 2 > 0$

Therefore, $\left(\dfrac{\pi}{2}, 1\right)$ is a relative minimum.

$f''\left(\dfrac{5\pi}{6}\right) = -3 < 0$

Therefore, $\left(\dfrac{5\pi}{6}, \dfrac{3}{2}\right)$ is a relative maximum.

$f''\left(\dfrac{3\pi}{2}\right) = 6 > 0$

Therefore, $\left(\dfrac{3\pi}{2}, -3\right)$ is a relative minimum.

43. $f(x) = 0.2x^2(x - 3)^3, [-1, 4]$

(a) $f'(x) = 0.2x(5x - 6)(x - 3)^2$

$f''(x) = (x - 3)(4x^2 - 9.6x + 3.6)$

$= 0.4(x - 3)(10x^2 - 24x + 9)$

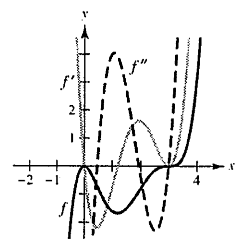
(b) $f''(0) < 0 \Rightarrow (0, 0)$ is a relative maximum.

$f''\left(\frac{6}{5}\right) > 0 \Rightarrow (1.2, -1.6796)$ is a relative minimum.

Points of inflection:

$(3, 0), (0.4652, -0.7048), (1.9348, -0.9049)$

(c)



f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

44. $f(x) = x^2\sqrt{6 - x^2}, \left[-\sqrt{6}, \sqrt{6}\right]$

(a) $f'(x) = \dfrac{3x(4 - x^2)}{\sqrt{6 - x^2}}$

$f'(x) = 0$ when $x = 0, x = \pm 2$.

$f''(x) = \dfrac{6(x^4 - 9x^2 + 12)}{(6 - x^2)^{3/2}}$

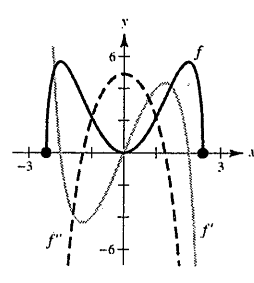
$f''(x) = 0$ when $x = \pm\sqrt{\dfrac{9 - \sqrt{33}}{2}}$.

(b) $f''(0) > 0 \Rightarrow (0, 0)$ is a relative minimum.

$f''(\pm 2) < 0 \Rightarrow (\pm 2, 4\sqrt{2})$ are relative maxima.

Points of inflection: $(\pm 1.2758, 3.4035)$

(c)



The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

45. $f(x) = \sin x - \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x, \quad [0, \pi]$

(a) $f'(x) = \cos x - \cos 3x + \cos 5x$

$f'(x) = 0$ when $x = \frac{\pi}{6}, x = \frac{\pi}{2}, x = \frac{5\pi}{6}$.

$f''(x) = -\sin x + 3\sin 3x - 5\sin 5x$

$f''(x) = 0$ when $x = \frac{\pi}{6}, x = \frac{5\pi}{6}$,

$x \approx 1.1731, x \approx 1.9685$

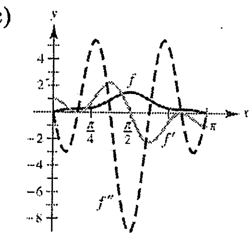
(b) $f''\left(\frac{\pi}{2}\right) < 0 \Rightarrow \left(\frac{\pi}{2}, 1.53333\right)$ is a relative maximum.

Points of inflection: $\left(\frac{\pi}{6}, 0.2667\right), (1.1731, 0.9638),$

$(1.9685, 0.9637), \left(\frac{5\pi}{6}, 0.2667\right)$

Note: $(0, 0)$ and $(\pi, 0)$ are not points of inflection because they are endpoints.

(c)



The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

46. $f(x) = \sqrt{2x}\sin x, [0, 2\pi]$

(a) $f'(x) = \sqrt{2x}\cos x + \frac{\sin x}{\sqrt{2x}}$

Critical numbers: $x \approx 1.84, 4.82$

$$f''(x) = -\sqrt{2x}\sin x + \frac{\cos x}{\sqrt{2x}} + \frac{\cos x}{\sqrt{2x}} - \frac{\sin x}{2x\sqrt{2x}}$$

$$= \frac{2\cos x}{\sqrt{2x}} - \frac{(4x^2 + 1)\sin x}{2x\sqrt{2x}}$$

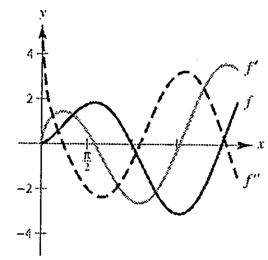
$$= \frac{4x\cos x - (4x^2 + 1)\sin x}{2x\sqrt{2x}}$$

(b) Relative maximum: $(1.84, 1.85)$

Relative minimum: $(4.82, -3.09)$

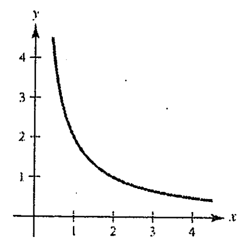
Points of inflection: $(0.75, 0.83), (3.42, -0.72)$

(c)



f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

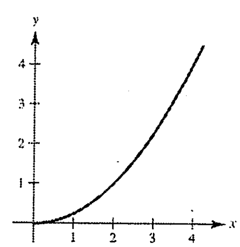
47. (a)



$f' < 0$ means f decreasing

f' increasing means concave upward

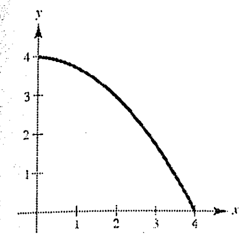
(b)



$f' > 0$ means f increasing

f' increasing means concave upward

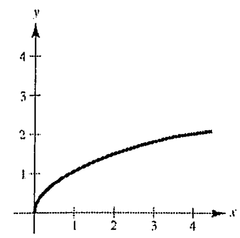
48. (a)



$f' < 0$ means f decreasing

f' decreasing means concave downward

(b)



$f' > 0$ means f increasing

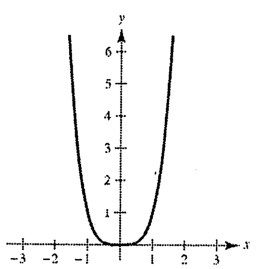
f' decreasing means concave downward

49. Answers will vary. *Sample answer*:

Let $f(x) = x^4$.

$f''(x) = 12x^2$

$f''(0) = 0$, but $(0, 0)$ is not a point of inflection.



50. (a) The rate of change of sales is increasing.

$S'' > 0$

(b) The rate of change of sales is decreasing.

$S' > 0, S'' < 0$

(c) The rate of change of sales is constant.

$S' = C, S'' = 0$

(d) Sales are steady.

$S = C, S' = 0, S'' = 0$

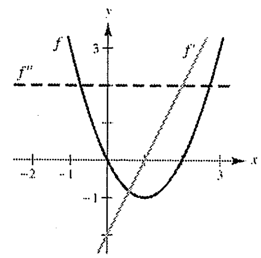
(e) Sales are declining, but at a lower rate.

$S' < 0, S'' > 0$

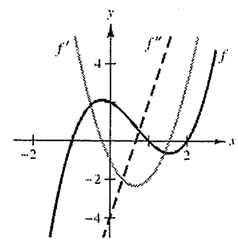
(f) Sales have bottomed out and have started to rise.

$S' > 0, S'' > 0$ Answers will vary.

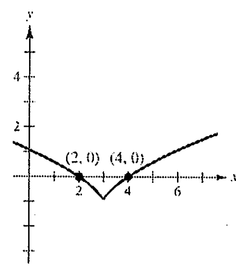
51.



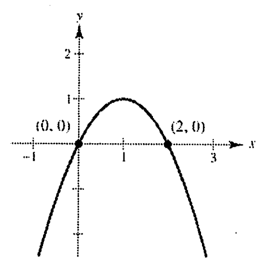
52.



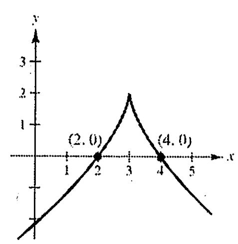
53.



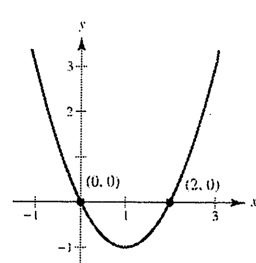
54.



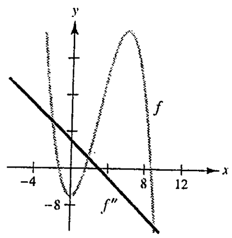
55.



56.



57.



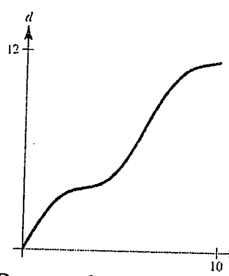
f'' is linear.

f' is quadratic.

f is cubic.

f concave upward on $(-\infty, 3)$, downward on $(3, \infty)$.

58. (a)



(b) Because the depth d is always increasing, there are no relative extrema. $f'(x) > 0$

(c) The rate of change of d is decreasing until you reach the widest point of the jug, then the rate increases until you reach the narrowest part of the jug's neck, then the rate decreases until you reach the top of the jug.

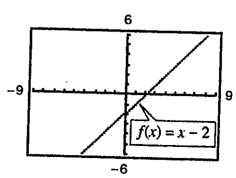
59. (a) $n = 1$:

$f(x) = x - 2$

$f'(x) = 1$

$f''(x) = 0$

No point of inflection



$n = 2$:

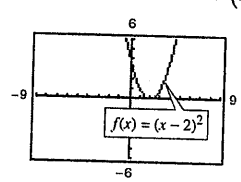
$f(x) = (x - 2)^2$

$f'(x) = 2(x - 2)$

$f''(x) = 2$

No point of inflection

Relative minimum: $(2, 0)$



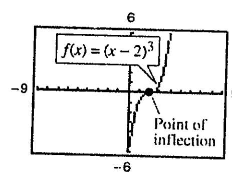
$n = 3$:

$f(x) = (x - 2)^3$

$f'(x) = 3(x - 2)^2$

$f''(x) = 6(x - 2)$

Point of inflection: $(2, 0)$



$n = 4$:

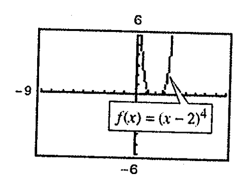
$f(x) = (x - 2)^4$

$f'(x) = 4(x - 2)^3$

$f''(x) = 12(x - 2)^2$

No point of inflection

Relative minimum: $(2, 0)$



Conclusion: If $n \geq 3$ and n is odd, then $(2, 0)$ is point of inflection. If $n \geq 2$ and n is even, then $(2, 0)$ is a relative minimum.

(b) Let $f(x) = (x - 2)^n, f'(x) = n(x - 2)^{n-1}, f''(x) = n(n - 1)(x - 2)^{n-2}$.

For $n \geq 3$ and odd, $n - 2$ is also odd and the concavity changes at $x = 2$.

For $n \geq 4$ and even, $n - 2$ is also even and the concavity does not change at $x = 2$.

So, $x = 2$ is point of inflection if and only if $n \geq 3$ is odd.

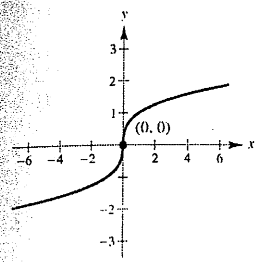
60. (a) $f(x) = \sqrt[3]{x}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f''(x) = -\frac{2}{9}x^{-5/3}$

Point of inflection: $(0, 0)$

(b) $f''(x)$ does not exist at $x = 0$.



61. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(3, 3)$

Relative minimum: $(5, 1)$

Point of inflection: $(4, 2)$

$f'(x) = 3ax^2 + 2bx + c,\ f''(x) = 6ax + 2b$

$$\left.\begin{aligned} f(3) &= 27a + 9b + 3c + d = 3 \\ f(5) &= 125a + 25b + 5c + d = 1 \end{aligned}\right\} 98a + 16b + 2c = -2 \Rightarrow 49a + 8b + c = -1$$

$f'(3) = 27a + 6b + c = 0,\ f''(4) = 24a + 2b = 0$

$$\begin{array}{ll} 49a + 8b + c = -1 & 24a + 2b = 0 \\ \underline{27a + 6b + c = 0} & \underline{22a + 2b = -1} \\ 22a + 2b = -1 & 2a = 1 \end{array}$$

$a = \frac{1}{2}, b = -6, c = \frac{45}{2}, d = -24$

$f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$

62. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(2, 4)$

Relative minimum: $(4, 2)$

Point of inflection: $(3, 3)$

$f'(x) = 3ax^2 + 2bx + c,\ f''(x) = 6ax + 2b$

$$\left.\begin{aligned} f(2) &= 8a + 4b + 2c + d = 4 \\ f(4) &= 64a + 16b + 4c + d = 2 \end{aligned}\right\} 56a + 12b + 2c = -2 \Rightarrow 28a + 6b + c = -1$$

$f'(2) = 12a + 4b + c = 0,\ f'(4) = 48a + 8b + c = 0,\ f''(3) = 18a + 2b = 0$

$$\begin{array}{ll} 28a + 6b + c = -1 & 18a + 2b = 0 \\ 12a + 4b + c = 0 & 16a + 2b = -1 \\ 16a + 2b = -1 & 2a = 1 \end{array}$$

$a = \frac{1}{2}, b = -\frac{9}{2}, c = 12, d = -6$

$f(x) = \frac{1}{2}x^3 - \frac{9}{2}x^2 + 12x - 6$

63. $f(x) = ax^3 + bx^2 + cx + d$

Maximum: $(-4, 1)$

Minimum: $(0, 0)$

(a) $f'(x) = 3ax^2 + 2bx + c, \qquad f''(x) = 6ax + 2b$

$$\begin{aligned} f(0) &= 0 \Rightarrow d = 0 \\ f(-4) &= 1 \Rightarrow -64a + 16b - 4c = 1 \\ f'(-4) &= 0 \Rightarrow 48a - 8b + c = 0 \\ f'(0) &= 0 \Rightarrow c = 0 \end{aligned}$$

Solving this system yields $a = \frac{1}{32}$ and $b = 6a = \frac{3}{16}$.

$f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$

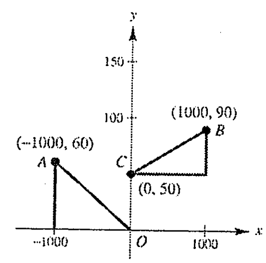
(b) The plane would be descending at the greatest rate at the point of inflection.

$f''(x) = 6ax + 2b = \frac{3}{16}x + \frac{3}{8} = 0 \Rightarrow x = -2.$

Two miles from touchdown.

64. (a) line OA: $y = -0.06x$ slope: -0.06

line CB: $y = 0.04x + 50$ slope: 0.04



$f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$(-1000, 60)$: $\quad 60 = (-1000)^3 a + (1000)^2 b - 1000c + d$

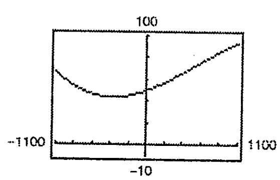
$\qquad -0.06 = (1000)^2\, 3a - 2000b + c$

$(1000, 90)$: $\quad 90 = (1000)^3 a + (1000)^2 b + 1000c + d$

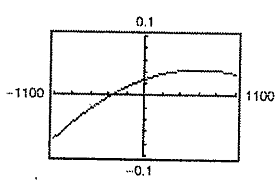
$\qquad 0.04 = (1000)^2\, 3a + 2000b + c$

The solution to this system of four equations is $a = -1.25 \times 10^{-8}$, $b = 0.000025$, $c = 0.0275$, and $d = 50$.

(b) $y = -1.25 \times 10^{-8}x^3 + 0.000025x^2 + 0.0275x + 50$



(c)



(d) The steepest part of the road is 6% at the point A.

65. $C = 0.5x^2 + 15x + 5000$

$$\overline{C} = \frac{C}{x} = 0.5x + 15 + \frac{5000}{x}$$

$\overline{C}$ = average cost per unit

$$\frac{d\overline{C}}{dx} = 0.5 - \frac{5000}{x^2} = 0 \text{ when } x = 100$$

By the First Derivative Test, $\overline{C}$ is minimized when $x = 100$ units.

66. $S = \dfrac{5.755}{10^8}T^3 - \dfrac{8.521}{10^6}T^2 + \dfrac{6.540}{10^5}T + 0.99987, 0 < T < 25$

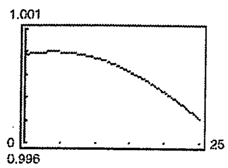
(a) $S' = \dfrac{17.265}{10^8}T^2 - \dfrac{17.042}{10^6}T + \dfrac{6.540}{10^5}$

$S'' = \dfrac{34.53}{10^8}T - \dfrac{17.042}{10^6} = 0$ when $T \approx 49.4$, which is not in the domain

$S'' < 0$ for $0 < T < 25 \Rightarrow$ Concave downward.

(b) The maximum is approximately $(4, 1)$.

(c)



(d) When $t = 20$, $S \approx 0.998$.

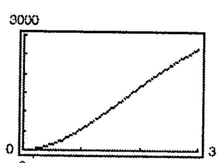
67. $S = \dfrac{5000t^2}{8 + t^2}, 0 \le t \le 3$

(a)

t	0.5	1	1.5	2	2.5	3
S	151.5	555.6	1097.6	1666.7	2193.0	2647.1

Increasing at greatest rate when $1.5 < t < 2$

(b)



Increasing at greatest rate when $t \approx 1.5$.

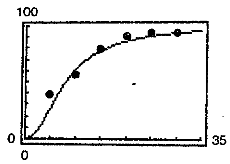
(c) $S = \dfrac{5000t^2}{8 + t^2}$

$$S'(t) = \frac{80{,}000t}{\left(8 + t^2\right)^2}$$

$$S''(t) = \frac{80{,}000\left(8 - 3t^2\right)}{\left(8 + t^2\right)^3}$$

$S''(t) = 0$ for $t = \pm\sqrt{\dfrac{8}{3}}$. So, $t = \dfrac{2\sqrt{6}}{3} \approx 1.633$ yrs.

68. $S = \dfrac{100t^2}{65 + t^2}, t > 0$

(a) 

(b) $S'(t) = \dfrac{13{,}000t}{\left(65 + t^2\right)^2}$

$$S''(t) = \frac{13{,}000\left(65 - 3t^2\right)}{\left(65 + t^2\right)^3} = 0 \Rightarrow t = 4.65$$

S is concave upwards on $(0, 4.65)$, concave downwards on $(4.65, 30)$.

(c) $S'(t) > 0$ for $t > 0$.

As t increases, the speed increases, but at a slower rate.

69. $f(x) = 2(\sin x + \cos x), \quad f\left(\dfrac{\pi}{4}\right) = 2\sqrt{2}$

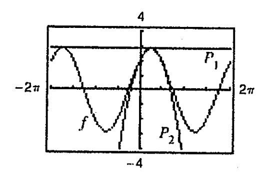
$f'(x) = 2(\cos x - \sin x), \quad f'\left(\dfrac{\pi}{4}\right) = 0$

$f''(x) = 2(-\sin x - \cos x), \quad f''\left(\dfrac{\pi}{4}\right) = -2\sqrt{2}$

$P_1(x) = 2\sqrt{2} + 0\left(x - \dfrac{\pi}{4}\right) = 2\sqrt{2}$

$P_1'(x) = 0$

$P_2(x) = 2\sqrt{2} + 0\left(x - \dfrac{\pi}{4}\right) + \dfrac{1}{2}\left(-2\sqrt{2}\right)\left(x - \dfrac{\pi}{4}\right)^2 = 2\sqrt{2} - \sqrt{2}\left(x - \dfrac{\pi}{4}\right)^2$



$P_2'(x) = -2\sqrt{2}\left(x - \dfrac{\pi}{4}\right)$

$P_2''(x) = -2\sqrt{2}$

The values of f, P_1, P_2, and their first derivatives are equal at $x = \pi/4$. The values of the second derivatives of f and P_2 are equal at $x = \pi/4$. The approximations worsen as you move away from $x = \pi/4$.

70. $f(x) = 2(\sin x + \cos x), \quad f(0) = 2$

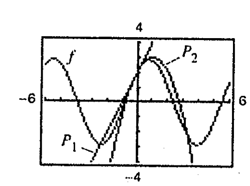
$f'(x) = 2(\cos x - \sin x), \quad f'(0) = 2$

$f''(x) = 2(-\sin x - \cos x), \quad f''(0) = -2$

$P_1(x) = 2 + 2(x - 0) = 2(1 + x)$

$P_1'(x) = 2$

$P_2(x) = 2 + 2(x - 0) + \tfrac{1}{2}(-2)(x - 0)^2 = 2 + 2x - x^2$



$P_2'(x) = 2 - 2x$

$P_2''(x) = -2$

The values of f, P_1, P_2, and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

71. $f(x) = \sqrt{1-x}, \qquad f(0) = 1$

$f'(x) = -\dfrac{1}{2\sqrt{1-x}}, \qquad f'(0) = -\dfrac{1}{2}$

$f''(x) = -\dfrac{1}{4(1-x)^{3/2}}, \qquad f''(0) = -\dfrac{1}{4}$

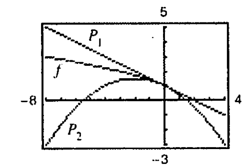
$P_1(x) = 1 + \left(-\dfrac{1}{2}\right)(x-0) = 1 - \dfrac{x}{2}$

$P_1'(x) = -\dfrac{1}{2}$

$P_2(x) = 1 + \left(-\dfrac{1}{2}\right)(x-0) + \dfrac{1}{2}\left(-\dfrac{1}{4}\right)(x-0)^2 = 1 - \dfrac{x}{2} - \dfrac{x^2}{8}$

$P_2'(x) = -\dfrac{1}{2} - \dfrac{x}{4}$

$P_2''(x) = -\dfrac{1}{4}$



The values of f, P_1, P_2, and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

72. $f(x) = \dfrac{\sqrt{x}}{x-1}, \qquad f(2) = \sqrt{2}$

$f'(x) = \dfrac{-(x+1)}{2\sqrt{x}(x-1)^2}, \qquad f'(2) = -\dfrac{3}{2\sqrt{2}} = -\dfrac{3\sqrt{2}}{4}$

$f''(x) = \dfrac{3x^2 + 6x - 1}{4x^{3/2}(x-1)^3}, \qquad f''(2) = \dfrac{23}{8\sqrt{2}} = \dfrac{23\sqrt{2}}{16}$

$P_1(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x-2) = -\dfrac{3\sqrt{2}}{4}x + \dfrac{5\sqrt{2}}{2}$

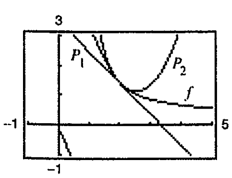
$P_1'(x) = -\dfrac{3\sqrt{2}}{4}$

$P_2(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x-2) + \dfrac{1}{2}\left(\dfrac{23\sqrt{2}}{16}\right)(x-2)^2 = \sqrt{2} - \dfrac{3\sqrt{2}}{4}(x-2) + \dfrac{23\sqrt{2}}{32}(x-2)^2$

$P_2'(x) = -\dfrac{3\sqrt{2}}{4} + \dfrac{23\sqrt{2}}{16}(x-2)$

$P_2''(x) = \dfrac{23\sqrt{2}}{16}$

The values of f, P_1, P_2 and their first derivatives are equal at $x = 2$. The values of the second derivatives of f and P_2 are equal at $x = 2$. The approximations worsen as you move away from $x = 2$.



73. $f(x) = x\sin\left(\frac{1}{x}\right)$

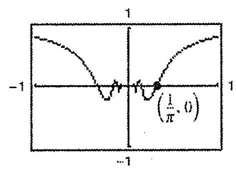
$$f'(x) = x\left[-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\right] + \sin\left(\frac{1}{x}\right) = -\frac{1}{x}\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

$$f''(x) = -\frac{1}{x}\left[\frac{1}{x^2}\sin\left(\frac{1}{x}\right)\right] + \frac{1}{x^2}\cos\left(\frac{1}{x}\right) - \frac{1}{x^2}\cos\left(\frac{1}{x}\right) = -\frac{1}{x^3}\sin\left(\frac{1}{x}\right) = 0$$

$$x = \frac{1}{\pi}$$

Point of inflection: $\left(\frac{1}{\pi}, 0\right)$

When $x > 1/\pi$, $f'' < 0$, so the graph is concave downward.



74. $f(x) = x(x - 6)^2 = x^3 - 12x^2 + 36x$

$f'(x) = 3x^2 - 24x + 36 = 3(x - 2)(x - 6) = 0$

$f''(x) = 6x - 24 = 6(x - 4) = 0$

Relative extrema: $(2, 32)$ and $(6, 0)$

Point of inflection $(4, 16)$ is midway between the relative extrema of f.

75. True. Let $y = ax^3 + bx^2 + cx + d, a \neq 0$. Then $y'' = 6ax + 2b = 0$ when $x = -(b/3a)$, and the concavity changes at this point.

76. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

77. False. Concavity is determined by f''. For example, let $f(x) = x$ and $c = 2$. $f'(c) = f'(2) > 0$, but f is not concave upward at $c = 2$.

78. False. For example, let $f(x) = (x - 2)^4$.

79. f and g are concave upward on (a, b) implies that f' and g' are increasing on (a, b), and $f'' > 0$ and $g'' > 0$.

So, $(f + g)'' > 0 \Rightarrow f + g$ is concave upward on (a, b) by Theorem 3.7.

80. f, g are positive, increasing, and concave upward on $(a, b) \Rightarrow f(x) > 0, f'(x) \geq 0$ and $f''(x) > 0$, and $g(x) > 0, g'(x) \geq 0$ and $g''(x) > 0$ on (a, b). For $x \in (a, b)$,

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$(fg)''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x) > 0$$

So, fg is concave upward on (a, b).

Section 3.5 Limits at Infinity

1. $f(x) = \frac{2x^2}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (f).

2. $f(x) = \frac{2x}{\sqrt{x^2 + 2}}$

No vertical asymptotes

Horizontal asymptotes: $y = \pm 2$

Matches (c).