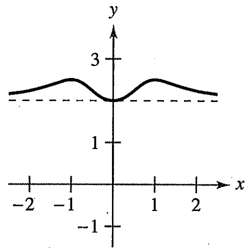
3.5 Exercises

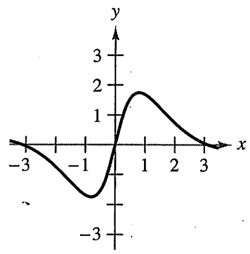
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Matching **In Exercises 1–6, match the function with one of the graphs [(a), (b), (c), (d), (e), or (f)] using horizontal asymptotes as an aid.**

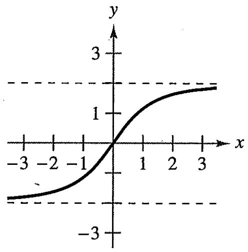
(a)



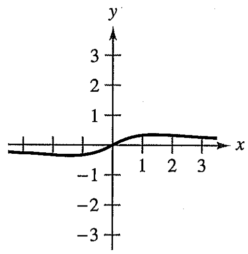
(b)



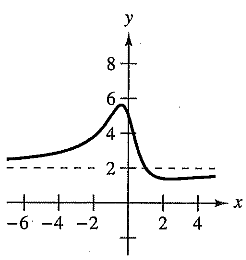
(c)



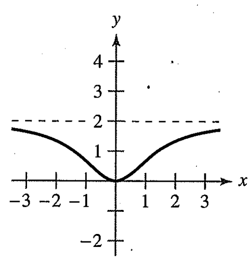
(d)



(e)



(f)



1. $f(x) = \dfrac{2x^2}{x^2 + 2}$

2. $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$

3. $f(x) = \dfrac{x}{x^2 + 2}$

4. $f(x) = 2 + \dfrac{x^2}{x^4 + 1}$

5. $f(x) = \dfrac{4 \sin x}{x^2 + 1}$

6. $f(x) = \dfrac{2x^2 - 3x + 5}{x^2 + 1}$

Numerical and Graphical Analysis **In Exercises 7–12, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit graphically.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

7. $f(x) = \dfrac{4x + 3}{2x - 1}$

8. $f(x) = \dfrac{2x^2}{x + 1}$

9. $f(x) = \dfrac{-6x}{\sqrt{4x^2 + 5}}$

10. $f(x) = \dfrac{10}{\sqrt{2x^2 - 1}}$

11. $f(x) = 5 - \dfrac{1}{x^2 + 1}$

12. $f(x) = 4 + \dfrac{3}{x^2 + 2}$

Finding Limits at Infinity **In Exercises 13 and 14, find $\lim_{x\to\infty} h(x)$, if possible.**

13. $f(x) = 5x^3 - 3x^2 + 10x$

(a) $h(x) = \dfrac{f(x)}{x^2}$

(b) $h(x) = \dfrac{f(x)}{x^3}$

(c) $h(x) = \dfrac{f(x)}{x^4}$

14. $f(x) = -4x^2 + 2x - 5$

(a) $h(x) = \dfrac{f(x)}{x}$

(b) $h(x) = \dfrac{f(x)}{x^2}$

(c) $h(x) = \dfrac{f(x)}{x^3}$

Finding Limits at Infinity **In Exercises 15–18, find each limit, if possible.**

15. (a) $\lim\limits_{x\to\infty} \dfrac{x^2 + 2}{x^3 - 1}$

(b) $\lim\limits_{x\to\infty} \dfrac{x^2 + 2}{x^2 - 1}$

(c) $\lim\limits_{x\to\infty} \dfrac{x^2 + 2}{x - 1}$

16. (a) $\lim\limits_{x\to\infty} \dfrac{3 - 2x}{3x^3 - 1}$

(b) $\lim\limits_{x\to\infty} \dfrac{3 - 2x}{3x - 1}$

(c) $\lim\limits_{x\to\infty} \dfrac{3 - 2x^2}{3x - 1}$

17. (a) $\lim\limits_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x^2 - 4}$

(b) $\lim\limits_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x^{3/2} - 4}$

(c) $\lim\limits_{x\to\infty} \dfrac{5 - 2x^{3/2}}{3x - 4}$

18. (a) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4x^2 + 1}$

(b) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4x^{3/2} + 1}$

(c) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4\sqrt{x} + 1}$

Finding a Limit **In Exercises 19–38, find the limit.**

19. $\lim\limits_{x\to\infty} \left(4 + \dfrac{3}{x}\right)$

20. $\lim\limits_{x\to-\infty} \left(\dfrac{5}{x} - \dfrac{x}{3}\right)$

21. $\lim\limits_{x\to\infty} \dfrac{2x - 1}{3x + 2}$

22. $\lim\limits_{x\to-\infty} \dfrac{4x^2 + 5}{x^2 + 3}$

23. $\lim\limits_{x\to\infty} \dfrac{x}{x^2 - 1}$

24. $\lim\limits_{x\to\infty} \dfrac{5x^3 + 1}{10x^3 - 3x^2 + 7}$

25. $\lim\limits_{x\to-\infty} \dfrac{5x^2}{x + 3}$

26. $\lim\limits_{x\to-\infty} \dfrac{x^3 - 4}{x^2 + 1}$

27. $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2 - x}}$

28. $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2 + 1}}$

29. $\lim\limits_{x\to-\infty} \dfrac{2x + 1}{\sqrt{x^2 - x}}$

30. $\lim\limits_{x\to\infty} \dfrac{5x^2 + 2}{\sqrt{x^2 + 3}}$

31. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2 - 1}}{2x - 1}$

32. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^4 - 1}}{x^3 - 1}$

33. $\lim\limits_{x\to\infty} \dfrac{x + 1}{(x^2 + 1)^{1/3}}$

34. $\lim\limits_{x\to-\infty} \dfrac{2x}{(x^6 - 1)^{1/3}}$

35. $\lim\limits_{x\to\infty} \dfrac{1}{2x + \sin x}$

36. $\lim\limits_{x\to\infty} \cos \dfrac{1}{x}$

37. $\lim\limits_{x\to\infty} \dfrac{\sin 2x}{x}$

38. $\lim\limits_{x\to\infty} \dfrac{x - \cos x}{x}$

Horizontal Asymptotes In Exercises 39–42, use a graphing utility to graph the function and identify any horizontal asymptotes.

39. $f(x) = \dfrac{|x|}{x+1}$
40. $f(x) = \dfrac{|3x+2|}{x-2}$
41. $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$
42. $f(x) = \dfrac{\sqrt{9x^2-2}}{2x+1}$

Finding a Limit In Exercises 43 and 44, find the limit. (*Hint:* Let $x = 1/t$ and find the limit as $t \to 0^+$.)

43. $\lim\limits_{x\to\infty} x \sin \dfrac{1}{x}$
44. $\lim\limits_{x\to\infty} x \tan \dfrac{1}{x}$

Finding a Limit In Exercises 45–48, find the limit. (*Hint:* Treat the expression as a fraction whose denominator is 1, and rationalize the numerator.) Use a graphing utility to verify your result.

45. $\lim\limits_{x\to-\infty} \left(x + \sqrt{x^2+3}\right)$
46. $\lim\limits_{x\to\infty} \left(x - \sqrt{x^2+x}\right)$
47. $\lim\limits_{x\to-\infty} \left(3x + \sqrt{9x^2-x}\right)$
48. $\lim\limits_{x\to\infty} \left(4x - \sqrt{16x^2-x}\right)$

Numerical, Graphical, and Analytic Analysis In Exercises 49–52, use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the function and estimate the limit. Finally, find the limit analytically and compare your results with the estimates.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

49. $f(x) = x - \sqrt{x(x-1)}$
50. $f(x) = x^2 - x\sqrt{x(x-1)}$
51. $f(x) = x \sin \dfrac{1}{2x}$
52. $f(x) = \dfrac{x+1}{x\sqrt{x}}$

WRITING ABOUT CONCEPTS

Writing In Exercises 53 and 54, describe in your own words what the statement means.

53. $\lim\limits_{x\to\infty} f(x) = 4$
54. $\lim\limits_{x\to-\infty} f(x) = 2$

55. **Sketching a Graph** Sketch a graph of a differentiable function f that satisfies the following conditions and has $x = 2$ as its only critical number.

$f'(x) < 0$ for $x < 2$

$f'(x) > 0$ for $x > 2$

$\lim\limits_{x\to-\infty} f(x) = 6$

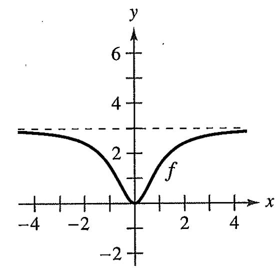
$\lim\limits_{x\to\infty} f(x) = 6$

56. **Points of Inflection** Is it possible to sketch a graph of a function that satisfies the conditions of Exercise 55 and has *no* points of inflection? Explain.

WRITING ABOUT CONCEPTS (continued)

57. **Using Symmetry to Find Limits** If f is a continuous function such that $\lim\limits_{x\to\infty} f(x) = 5$, find, if possible, $\lim\limits_{x\to-\infty} f(x)$ for each specified condition.
 (a) The graph of f is symmetric with respect to the y-axis.
 (b) The graph of f is symmetric with respect to the origin.

58. **A Function and Its Derivative** The graph of a function f is shown below. To print an enlarged copy of the graph, go to *MathGraphs.com*.



 (a) Sketch f'.
 (b) Use the graphs to estimate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} f'(x)$.
 (c) Explain the answers you gave in part (b).

Sketching a Graph In Exercises 59–74, sketch the graph of the equation using extrema, intercepts, symmetry, and asymptotes. Then use a graphing utility to verify your result.

59. $y = \dfrac{x}{1-x}$
60. $y = \dfrac{x-4}{x-3}$
61. $y = \dfrac{x+1}{x^2-4}$
62. $y = \dfrac{2x}{9-x^2}$
63. $y = \dfrac{x^2}{x^2+16}$
64. $y = \dfrac{2x^2}{x^2-4}$
65. $xy^2 = 9$
66. $x^2y = 9$
67. $y = \dfrac{3x}{x-1}$
68. $y = \dfrac{3x}{1-x^2}$
69. $y = 2 - \dfrac{3}{x^2}$
70. $y = 1 - \dfrac{1}{x}$
71. $y = 3 + \dfrac{2}{x}$
72. $y = \dfrac{4}{x^2} + 1$
73. $y = \dfrac{x^3}{\sqrt{x^2-4}}$
74. $y = \dfrac{x}{\sqrt{x^2-4}}$

Analyzing a Graph Using Technology In Exercises 75–82, use a computer algebra system to analyze the graph of the function. Label any extrema and/or asymptotes that exist.

75. $f(x) = 9 - \dfrac{5}{x^2}$
76. $f(x) = \dfrac{1}{x^2-x-2}$
77. $f(x) = \dfrac{x-2}{x^2-4x+3}$
78. $f(x) = \dfrac{x+1}{x^2+x+1}$

79. $f(x) = \dfrac{3x}{\sqrt{4x^2 + 1}}$

80. $g(x) = \dfrac{2x}{\sqrt{3x^2 + 1}}$

81. $g(x) = \sin\left(\dfrac{x}{x - 2}\right), \quad x > 3$

82. $f(x) = \dfrac{2 \sin 2x}{x}$

Comparing Functions **In Exercises 83 and 84, (a) use a graphing utility to graph f and g in the same viewing window, (b) verify algebraically that f and g represent the same function, and (c) zoom out sufficiently far so that the graph appears as a line. What equation does this line appear to have? (Note that the points at which the function is not continuous are not readily seen when you zoom out.)**

83. $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x - 3)}$

$g(x) = x + \dfrac{2}{x(x - 3)}$

84. $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$

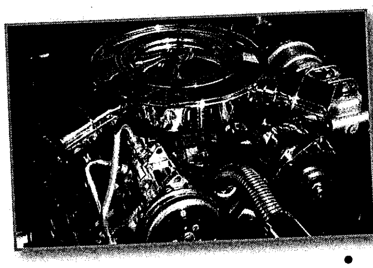
$g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

85. Engine Efficiency

The efficiency of an internal combustion engine is

$$\text{Efficiency (\%)} = 100\left[1 - \frac{1}{(v_1/v_2)^c}\right]$$

where v_1/v_2 is the ratio of the uncompressed gas to the compressed gas and c is a positive constant dependent on the engine design. Find the limit of the efficiency as the compression ratio approaches infinity.

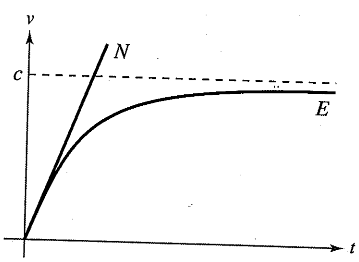


86. Average Cost A business has a cost of $C = 0.5x + 500$ for producing x units. The average cost per unit is

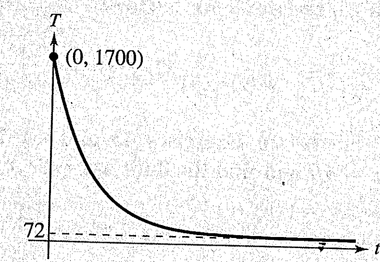
$$\overline{C} = \frac{C}{x}.$$

Find the limit of $\overline{C}$ as x approaches infinity.

87. Physics Newton's First Law of Motion and Einstein's Special Theory of Relativity differ concerning a particle's behavior as its velocity approaches the speed of light c. In the graph, functions N and E represent the velocity v, with respect to time t, of a particle accelerated by a constant force as predicted by Newton and Einstein, respectively. Write limit statements that describe these two theories.



88. HOW DO YOU SEE IT? The graph shows the temperature T, in degrees Fahrenheit, of molten glass t seconds after it is removed from a kiln.



(a) Find $\lim_{t \to 0^+} T$. What does this limit represent?

(b) Find $\lim_{t \to \infty} T$. What does this limit represent?

(c) Will the temperature of the glass ever actually reach room temperature? Why?

89. Modeling Data The average typing speeds S (in words per minute) of a typing student after t weeks of lessons are shown in the table.

t	5	10	15	20	25	30
S	28	56	79	90	93	94

A model for the data is $S = \dfrac{100t^2}{65 + t^2}, \; t > 0$.

(a) Use a graphing utility to plot the data and graph the model.

(b) Does there appear to be a limiting typing speed? Explain.

90. Modeling Data A heat probe is attached to the heat exchanger of a heating system. The temperature T (in degrees Celsius) is recorded t seconds after the furnace is started. The results for the first 2 minutes are recorded in the table.

t	0	15	30	45	60
T	25.2°	36.9°	45.5°	51.4°	56.0°

t	75	90	105	120
T	59.6°	62.0°	64.0°	65.2°

(a) Use the regression capabilities of a graphing utility to find a model of the form $T_1 = at^2 + bt + c$ for the data.

(b) Use a graphing utility to graph T_1.

(c) A rational model for the data is

$$T_2 = \frac{1451 + 86t}{58 + t}.$$

Use a graphing utility to graph T_2.

(d) Find $T_1(0)$ and $T_2(0)$.

(e) Find $\lim_{t \to \infty} T_2$.

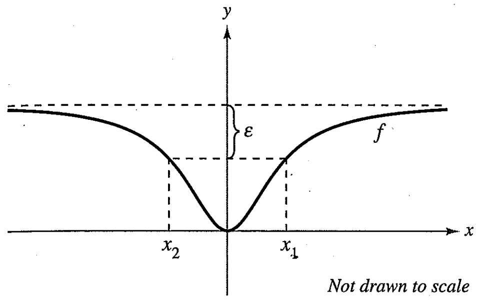
(f) Interpret the result in part (e) in the context of the problem. Is it possible to do this type of analysis using T_1? Explain.

Straight 8 Photography/Shutterstock.com

91. **Using the Definition of Limits at Infinity** The graph of

$$f(x) = \frac{2x^2}{x^2 + 2}$$

is shown.



(a) Find $L = \lim_{x\to\infty} f(x)$.

(b) Determine x_1 and x_2 in terms of ε.

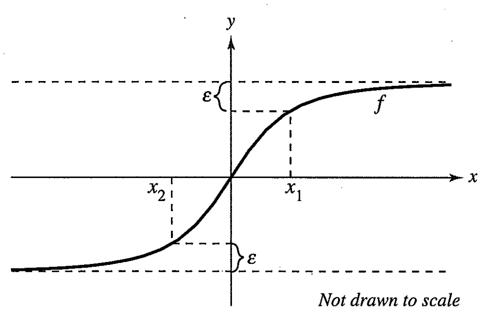
(c) Determine M, where $M > 0$, such that $|f(x) - L| < \varepsilon$ for $x > M$.

(d) Determine N, where $N < 0$, such that $|f(x) - L| < \varepsilon$ for $x < N$.

92. **Using the Definition of Limits at Infinity** The graph of

$$f(x) = \frac{6x}{\sqrt{x^2 + 2}}$$

is shown.



(a) Find $L = \lim_{x\to\infty} f(x)$ and $K = \lim_{x\to-\infty} f(x)$.

(b) Determine x_1 and x_2 in terms of ε.

(c) Determine M, where $M > 0$, such that $|f(x) - L| < \varepsilon$ for $x > M$.

(d) Determine N, where $N < 0$, such that $|f(x) - K| < \varepsilon$ for $x < N$.

93. **Using the Definition of Limits at Infinity** Consider

$$\lim_{x\to\infty} \frac{3x}{\sqrt{x^2 + 3}}.$$

(a) Use the definition of limits at infinity to find values of M that correspond to $\varepsilon = 0.5$.

(b) Use the definition of limits at infinity to find values of M that correspond to $\varepsilon = 0.1$.

94. **Using the Definition of Limits at Infinity** Consider

$$\lim_{x\to-\infty} \frac{3x}{\sqrt{x^2 + 3}}.$$

(a) Use the definition of limits at infinity to find values of N that correspond to $\varepsilon = 0.5$.

(b) Use the definition of limits at infinity to find values of N that correspond to $\varepsilon = 0.1$.

Proof In Exercises 95–98, use the definition of limits at infinity to prove the limit.

95. $\lim_{x\to\infty} \frac{1}{x^2} = 0$

96. $\lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$

97. $\lim_{x\to-\infty} \frac{1}{x^3} = 0$

98. $\lim_{x\to-\infty} \frac{1}{x - 2} = 0$

99. **Distance** A line with slope m passes through the point $(0, 4)$.

(a) Write the shortest distance d between the line and the point $(3, 1)$ as a function of m.

(b) Use a graphing utility to graph the equation in part (a).

(c) Find $\lim_{m\to\infty} d(m)$ and $\lim_{m\to-\infty} d(m)$. Interpret the results geometrically.

100. **Distance** A line with slope m passes through the point $(0, -2)$.

(a) Write the shortest distance d between the line and the point $(4, 2)$ as a function of m.

(b) Use a graphing utility to graph the equation in part (a).

(c) Find $\lim_{m\to\infty} d(m)$ and $\lim_{m\to-\infty} d(m)$. Interpret the results geometrically.

101. **Proof** Prove that if

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

and

$$q(x) = b_m x^m + \cdots + b_1 x + b_0$$

where $a_n \neq 0$ and $b_m \neq 0$, then

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m. \\ \pm\infty, & n > m \end{cases}$$

102. **Proof** Use the definition of infinite limits at infinity to prove that $\lim_{x\to\infty} x^3 = \infty$.

True or False? In Exercises 103 and 104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

103. If $f'(x) > 0$ for all real numbers x, then f increases without bound.

104. If $f''(x) < 0$ for all real numbers x, then f decreases without bound.