101. $\lim_{x\to\infty}\frac{p(x)}{q(x)} = \lim_{x\to\infty}\frac{a_nx^n + \cdots + a_1x + a_0}{b_mx^m + \cdots + b_1x + b_0}$

Divide $p(x)$ and $q(x)$ by x^m.

Case 1: If $n < m$: $\lim_{x\to\infty}\frac{p(x)}{q(x)} = \lim_{x\to\infty}\frac{\frac{a_n}{x^{m-n}} + \cdots + \frac{a_1}{x^{m-1}} + \frac{a_0}{x^m}}{b_m + \cdots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}} = \frac{0 + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \frac{0}{b_m} = 0.$

Case 2: If $m = n$: $\lim_{x\to\infty}\frac{p(x)}{q(x)} = \lim_{x\to\infty}\frac{a_n + \cdots + \frac{a_1}{x^{m-1}} + \frac{a_0}{x^m}}{b_m + \cdots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}} = \frac{a_n + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \frac{a_n}{b_m}.$

Case 3: If $n > m$: $\lim_{x\to\infty}\frac{p(x)}{q(x)} = \lim_{x\to\infty}\frac{a_nx^{n-m} + \cdots + \frac{a_1}{x^{m-1}} + \frac{a_0}{x^m}}{b_m + \cdots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}} = \frac{\pm\infty + \cdots + 0}{b_m + \cdots + 0} = \pm\infty.$

102. $\lim_{x\to\infty} x^3 = \infty$. Let $M > 0$ be given. You need $N > 0$ such that $f(x) = x^3 > M$ whenever $x > N$.

$x^3 > M \Rightarrow x > M^{1/3}$. Let $N = M^{1/3}$. For $x > N = M^{1/3}, x > M^{1/3} \Rightarrow x^3 > M \Rightarrow f(x) > M.$

103. False. Let $f(x) = \frac{2x}{\sqrt{x^2 + 2}}$. (See Exercise 54.)

104. False. Let $y_1 = \sqrt{x + 1}$, then $y_1(0) = 1$. So $y_1' = 1/\left(2\sqrt{x + 1}\right)$ and $y_1'(0) = 1/2$. Finally, $y_1'' = -\frac{1}{4(x + 1)^{3/2}}$ and

$y_1''(0) = -\frac{1}{4}$. Let $p = ax^2 + bx + 1$, then $p(0) = 1$. So, $p' = 2ax + b$ and $p'(0) = \frac{1}{2} \Rightarrow b = \frac{1}{2}$. Finally, $p'' = 2a$ and $p''(0) = -\frac{1}{4} \Rightarrow a = -\frac{1}{8}$. Therefore,

$$f(x) = \begin{cases}(-1/8)x^2 + (1/2)x + 1, & x < 0\\ \sqrt{x + 1}, & x \ge 0\end{cases} \text{ and } f(0) = 1,$$

$$f'(x) = \begin{cases}(1/2) - (1/4)x, & x < 0\\ 1/\left(2\sqrt{x + 1}\right), & x > 0\end{cases} \text{ and } f'(0) = \frac{1}{2}, \text{ and}$$

$$f''(x) = \begin{cases}(-1/4), & x < 0\\ -1/\left(4(x + 1)^{3/2}\right), & x > 0\end{cases} \text{ and } f''(0) = -\frac{1}{4}.$$

$f''(x) < 0$ for all real x, but $f(x)$ *increases* without bound.

Section 3.6 A Summary of Curve Sketching

1. f has constant negative slope. Matches (d)

2. The slope of f approaches ∞ as $x \to 0^-$, and approaches $-\infty$ as $x \to 0^+$. Matches (c)

3. The slope is periodic, and zero at $x = 0$. Matches (a)

4. The slope is positive up to approximately $x = 1.5$. Matches (b)

5. $y = \dfrac{1}{x-2} - 3$

$y' = -\dfrac{1}{(x-2)^2} \Rightarrow$ undefined when $x = 2$

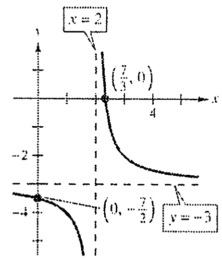
$y'' = \dfrac{2}{(x-2)^3} \Rightarrow$ undefined when $x = 2$

Intercepts: $\left(\dfrac{7}{3}, 0\right), \left(0, -\dfrac{7}{2}\right)$

Vertical asymptote: $x = 2$

Horizontal asymptote: $y = -3$

	y	y'	y''	Conclusion
$-\infty < x < 2$		$-$	$-$	Decreasing, concave down
$2 < x < \infty$		$-$	$+$	Decreasing, concave up



No relative extrema, no points of inflection

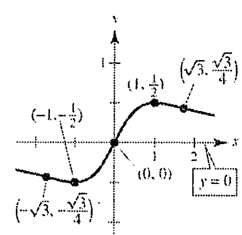
6. $y = \dfrac{x}{x^2+1}$

$y' = \dfrac{1-x^2}{\left(x^2+1\right)^2} = \dfrac{(1-x)(x+1)}{\left(x^2+1\right)^2} = 0$ when $x = \pm 1$.

$y'' = -\dfrac{2x\left(3-x^2\right)}{\left(x^2+1\right)^3} = 0$ when $x = 0, \pm\sqrt{3}$.

Horizontal asymptote: $y = 0$

	y	y'	y''	Conclusion
$-\infty < x < -\sqrt{3}$		$-$	$-$	Decreasing, concave down
$x = -\sqrt{3}$	$-\dfrac{\sqrt{3}}{4}$	$-$	0	Point of inflection
$-\sqrt{3} < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-\dfrac{1}{2}$	0	$+$	Relative minimum
$-1 < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	0	$+$	0	Point of inflection
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	$\dfrac{1}{2}$	0	$-$	Relative maximum
$1 < x < \sqrt{3}$		$-$	$-$	Decreasing, concave down
$x = \sqrt{3}$	$\dfrac{\sqrt{3}}{4}$	$-$	0	Point of inflection
$\sqrt{3} < x < \infty$		$-$	$+$	Decreasing, concave up



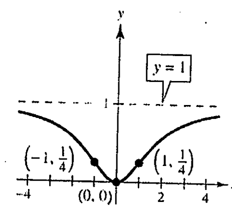
7. $y = \dfrac{x^2}{x^2 + 3}$

$y' = \dfrac{6x}{(x^2 + 3)^2} = 0$ when $x = 0$.

$y'' = \dfrac{18(1 - x^2)}{(x^2 + 3)^3} = 0$ when $x = \pm 1$.

Horizontal asymptote: $y = 1$

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$-$	Decreasing, concave down
$x = -1$	$\frac{1}{4}$	$-$	0	Point of inflection
$-1 < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	0	0	$+$	Relative minimum
$0 < x < 1$		$+$	$+$	Increasing, concave up
$x = 1$	$\frac{1}{4}$	$+$	0	Point of inflection
$1 < x < \infty$		$+$	$-$	Increasing, concave down



8. $y = \dfrac{x^2 + 1}{x^2 - 4}$

$y' = \dfrac{-10x}{(x^2 - 4)^2} = 0$ when $x = 0$ and undefined when $x = \pm 2$.

$y'' = \dfrac{10(3x^2 + 4)}{(x^2 - 4)^3} < 0$ when $x = 0$.

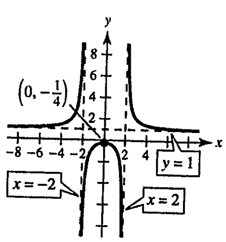
Intercept: $(0, -1/4)$

Symmetric about y-axis

Vertical asymptotes: $x = \pm 2$

Horizontal asymptote: $y = 1$

	y	y'	y''	Conclusion
$-\infty < x < -2$		$+$	$+$	Increasing, concave up
$-2 < x < 0$		$+$	$-$	Increasing, concave down
$x = 0$	$-\frac{1}{4}$			Relative maximum
$0 < x < 2$		$-$	$-$	Decreasing, concave down
$2 < x < \infty$		$-$	$+$	Decreasing, concave up



9. $y = \dfrac{3x}{x^2 - 1}$

$y' = \dfrac{-3(x^2 + 1)}{(x^2 - 1)^2}$ undefined when $x = \pm 1$

$y'' = \dfrac{6x(x^2 + 3)}{(x^2 - 1)^3}$

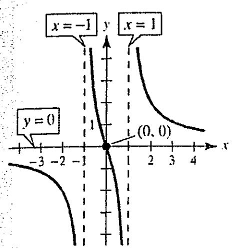
Intercept: $(0, 0)$

Symmetry with respect to origin

Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 0$

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$-$	Decreasing, concave down
$-1 < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	0	-3	0	Point of inflection
$0 < x < 1$		$-$	$-$	Decreasing, concave down
$1 < x < \infty$		$-$	$+$	Decreasing, concave up



10. $f(x) = \dfrac{x - 3}{x} = 1 - \dfrac{3}{x}$

$f'(x) = \dfrac{3}{x^2}$ undefined when $x = 0$

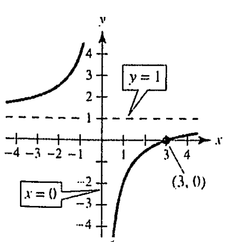
$f''(x) = -\dfrac{6}{x^3} \neq 0$

Vertical asymptote: $x = 0$

Intercept: $(3, 0)$

Horizontal asymptote: $y = 1$

	y	y'	y''	Conclusion
$-\infty < x < 0$		$+$	$+$	Increasing, concave up
$0 < x < \infty$		$+$	$-$	Increasing, concave down



11. $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3} = \dfrac{(x-4)(x^2+4x+16)}{x^3} = 0$ when $x = 4$ and undefined when $x = 0$.

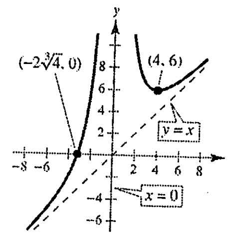
$f''(x) = \dfrac{192}{x^4}$

Intercept: $\left(-2\sqrt[3]{4}, 0\right)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$

	y	y'	y''	Conclusion
$-\infty < x < 0$		+	+	Increasing, concave up
$0 < x < 4$		−	+	Decreasing, concave up
$x = 4$	6	0	+	Relative minimum
$4 < x < \infty$		+	+	Increasing, concave up



12. $f(x) = \dfrac{x^3}{x^2-9} = x + \dfrac{9x}{x^2-9}$

$f'(x) = \dfrac{x^2(x^2-27)}{(x^2-9)^2} = 0$ when $x = 0,\ \pm 3\sqrt{3}$ and is undefined when $x = \pm 3$.

$f''(x) = \dfrac{18x(x^2+27)}{(x^2-9)^3} = 0$ when $x = 0$

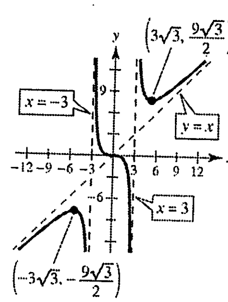
Intercept: $(0, 0)$

Symmetry: origin

Vertical asymptotes: $x = \pm 3$

Slant asymptote: $y = x$

	y	y'	y''	Conclusion
$-\infty < x < -3\sqrt{3}$		+	−	Increasing, concave down
$x = -3\sqrt{3}$	$-\dfrac{9\sqrt{3}}{2}$	0	−	Relative maximum
$-3\sqrt{3} < x < -3$		−	−	Decreasing, concave down
$-3 < x < 0$		−	+	Decreasing, concave up
$x = 0$	0	0	0	Point of inflection
$0 < x < 3$		−	−	Decreasing, concave down
$3 < x < 3\sqrt{3}$		−	+	Decreasing, concave up
$x = 3\sqrt{3}$	$\dfrac{9\sqrt{3}}{2}$	0	+	Relative minimum
$3\sqrt{3} < x < \infty$		+	+	Increasing, concave up



13. $y = \dfrac{x^2 - 6x + 12}{x - 4} = x - 2 + \dfrac{4}{x - 4}$

$y' = 1 - \dfrac{4}{(x-4)^2}$

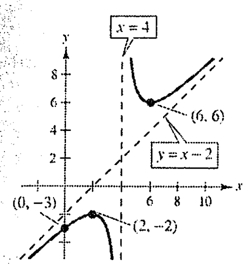
$= \dfrac{(x-2)(x-6)}{(x-4)^2} = 0$ when $x = 2, 6$ and is undefined when $x = 4$.

$y'' = \dfrac{8}{(x-4)^3}$

Vertical asymptote: $x = 4$

Slant asymptote: $y = x - 2$

	y	y'	y''	Conclusion
$-\infty < x < 2$		+	−	Increasing, concave down
$x = 2$	−2	0	−	Relative maximum
$2 < x < 4$		−	−	Decreasing, concave down
$4 < x < 6$		−	+	Decreasing, concave up
$x = 6$	6	0	+	Relative minimum
$6 < x < \infty$		+	+	Increasing, concave up



14. $y = \dfrac{-x^2 - 4x - 7}{x + 3} = -x - 1 - \dfrac{4}{x + 3}$

$y' = -\dfrac{x^2 + 6x + 5}{(x+3)^2} = -\dfrac{(x+1)(x+5)}{(x+3)^2} = 0$ when $x = -1, -5$ and is undefined when $x = -3$.

$y'' = \dfrac{-8}{(x+3)^3}$

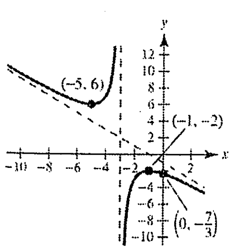
Intercept: $\left(0, -\frac{7}{3}\right)$

No symmetry

Vertical asymptote: $x = -3$

Slant asymptote: $y = -x - 1$

	y	y'	y''	Conclusion
$-\infty < x < -5$		−	+	Decreasing, concave up
$x = -5$	6	0	+	Relative minimum
$-5 < x < -3$		+	+	Increasing, concave up
$-3 < x < -1$		+	−	Increasing, concave down
$x = -1$	−2	0	−	Relative maximum
$-1 < x < \infty$		−	−	Decreasing, concave down

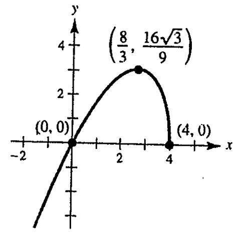


15. $y = x\sqrt{4-x}$, Domain: $(-\infty, 4]$

$$y' = \frac{8-3x}{2\sqrt{4-x}} = 0 \text{ when } x = \frac{8}{3} \text{ and undefined when } x = 4.$$

$$y'' = \frac{3x-16}{4(4-x)^{3/2}} = 0 \text{ when } x = \frac{16}{3} \text{ and undefined when } x = 4.$$

Note: $x = \frac{16}{3}$ is not in the domain.



	y	y'	y''	Conclusion
$-\infty < x < \frac{8}{3}$		+	−	Increasing, concave down
$x = \frac{8}{3}$	$\frac{16}{3\sqrt{3}}$	0	−	Relative maximum
$\frac{8}{3} < x < 4$		−	−	Decreasing, concave down
$x = 4$	0	Undefined	Undefined	Endpoint

16. $h(x) = x\sqrt{9-x^2}$, Domain: $-3 \le x \le 3$

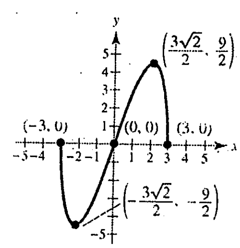
$$h'(x) = \frac{9-2x^2}{\sqrt{9-x^2}} = 0 \text{ when } x = \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2} \text{ and undefined when } x = \pm 3.$$

$$h''(x) = \frac{x(2x^2-27)}{(9-x^2)^{3/2}} = 0 \text{ when } x = 0 \text{ and undefined when } x = \pm 3.$$

Intercepts: $(0, 0)$, $(\pm 3, 0)$

Symmetric with respect to the origin

	y	y'	y''	Conclusion
$x = -3$	0	Undefined	Undefined	Endpoint
$-3 < x < -\frac{3}{\sqrt{2}}$		−	+	Decreasing, concave up
$x = -\frac{3}{\sqrt{2}}$	$-\frac{9}{2}$	0	+	Relative minimum
$-\frac{3}{\sqrt{2}} < x < 0$		+	+	Increasing, concave up
$x = 0$	0	3	0	Point of inflection
$0 < x < \frac{3}{\sqrt{2}}$		+	−	Increasing, concave down
$x = \frac{3}{\sqrt{2}}$	$\frac{9}{2}$	0	−	Relative maximum
$\frac{3}{\sqrt{2}} < x < 3$		−	−	Decreasing, concave down
$x = 3$	0	Undefined	Undefined	Endpoint



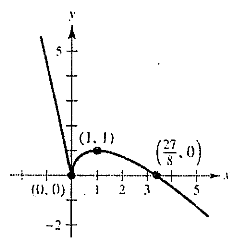
17. $y = 3x^{2/3} - 2x$

$y' = 2x^{-1/3} - 2 = \dfrac{2(1 - x^{1/3})}{x^{1/3}}$

$= 0$ when $x = 1$ and undefined when $x = 0$.

$y'' = \dfrac{-2}{3x^{4/3}} < 0$ when $x \neq 0$.

	y	y'	y''	Conclusion
$-\infty < x < 0$		$-$	$-$	Decreasing, concave down
$x = 0$	0	Undefined	Undefined	Relative minimum
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	1	0	$-$	Relative maximum
$1 < x < \infty$		$-$	$-$	Decreasing, concave down



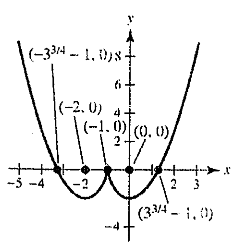
18. $y = (x + 1)^2 - 3(x + 1)^{2/3}$

$y' = 2(x + 1) - 2(x + 1)^{-1/3} = \dfrac{2(x + 1)^{4/3} - 2}{(x + 1)^{1/3}} = 0$ when $x = 0, -2$ and undefined when $x = -1$.

$y'' = 2 + \dfrac{2}{3}(x + 1)^{-4/3} = \dfrac{6(x + 1)^{4/3} + 2}{3(x + 1)^{4/3}}$

Intercepts: $(-1, 0), (\pm 3^{3/4} - 1, 0)$

	y	y'	y''	Conclusion
$-\infty < x < -2$		$-$	$+$	Decreasing, concave up
$x = -2$	-2	0	$+$	Relative minimum
$-2 < x < -1$		$+$	$+$	Increasing, concave up
$x = -1$	0	Undefined	$+$	Relative maximum
$-1 < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	0	0	$+$	Relative minimum
$0 < x < \infty$		$+$	$+$	Increasing, concave up



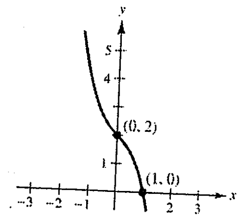
19. $y = 2 - x - x^3$

$y' = -1 - 3x^2$

No critical numbers

$y'' = -6x = 0$ when $x = 0$.

	y	y'	y''	Conclusion
$-\infty < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	2	$-$	0	Point of inflection
$0 < x < \infty$		$-$	$-$	Decreasing, concave down

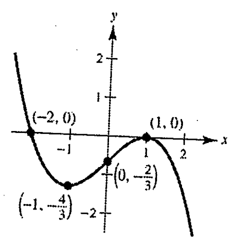


20. $y = -\frac{1}{3}(x^3 - 3x + 2)$

$y' = -x^2 + 1 = 0$ when $x = \pm 1$.

$y'' = -2x = 0$ when $x = 0$.

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-\frac{4}{3}$	0	$+$	Relative minimum
$-1 < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	$-\frac{2}{3}$	$+$	0	Point of inflection
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	0	0	$-$	Relative maximum
$1 < x < \infty$		$-$	$-$	Decreasing, concave down

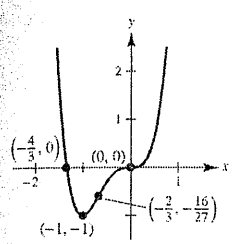


21. $y = 3x^4 + 4x^3$

$y' = 12x^3 + 12x^2 = 12x^2(x + 1) = 0$ when $x = 0, x = -1$.

$y'' = 36x^2 + 24x = 12x(3x + 2) = 0$ when $x = 0, x = -\frac{2}{3}$.

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	-1	0	$+$	Relative minimum
$-1 < x < -\frac{2}{3}$		$+$	$+$	Increasing, concave up
$x = -\frac{2}{3}$	$-\frac{16}{27}$	$+$	0	Point of inflection
$-\frac{2}{3} < x < 0$		$+$	$-$	Increasing, concave down
$x = 0$	0	0	0	Point of inflection
$0 < x < \infty$		$+$	$+$	Increasing, concave up



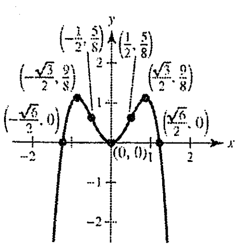
22. $y = -2x^4 + 3x^2$

$y' = -8x^3 + 6x = 0$ when $x = 0, \pm\frac{\sqrt{3}}{2}$.

$y'' = -24x^2 + 6 = 0$ when $x = \pm\frac{1}{2}$.

Symmetry: y-axis

Intercepts: $\left(\pm\frac{\sqrt{6}}{2}, 0\right)$



	y	y'	y''	Conclusion
$-\infty < x < -\frac{\sqrt{3}}{2}$		$+$	$-$	Increasing, concave down
$x = -\frac{\sqrt{3}}{2}$	$\frac{9}{8}$	0	$-$	Relative maximum
$-\frac{\sqrt{3}}{2} < x < -\frac{1}{2}$		$-$	$-$	Decreasing, concave down
$x = -\frac{1}{2}$	$\frac{5}{8}$	-2	0	Point of inflection
$-\frac{1}{2} < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	0	0	$+$	Relative minimum
$0 < x < \frac{1}{2}$		$+$	$+$	Increasing, concave up
$x = \frac{1}{2}$	$\frac{5}{8}$	2	0	Point of inflection
$\frac{1}{2} < x < \frac{\sqrt{3}}{2}$		$+$	$-$	Increasing, concave down
$x = \frac{\sqrt{3}}{2}$	$\frac{9}{8}$	0	$-$	Relative maximum
$\frac{\sqrt{3}}{2} < x < \infty$		$-$	$-$	Decreasing, concave down