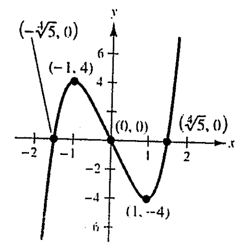
23. $y = x^5 - 5x$

$y' = 5x^4 - 5 = 5(x^4 - 1) = 0$ when $x = \pm 1$.

$y'' = 20x^3 = 0$ when $x = 0$.

	y	y'	y''	Conclusion
$-\infty < x < -1$		+	−	Increasing, concave down
$x = -1$	4	0	−	Relative maximum
$-1 < x < 0$		−	−	Decreasing, concave down
$x = 0$	0	−	0	Point of inflection
$0 < x < 1$		−	+	Decreasing, concave up
$x = 1$	−4	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up

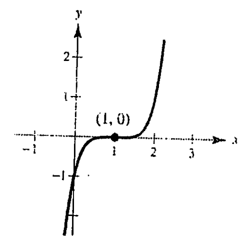


24. $y = (x - 1)^5$

$y' = 5(x - 1)^4 = 0$ when $x = 1$.

$y'' = 20(x - 1)^3 = 0$ when $x = 1$.

	y	y'	y''	Conclusion
$-\infty < x < 1$		+	−	Increasing, concave down
$x = 1$	0	0	0	Point of inflection
$1 < x < \infty$		+	+	Increasing, concave up



25. $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x} = \dfrac{19x^2 - 1}{x(x^2 + 1)}$

$f'(x) = \dfrac{-(19x^4 - 22x^2 - 1)}{x^2(x^2 + 1)^2} = 0$ for $x \approx \pm 1.10$

$f''(x) = \dfrac{2(19x^6 - 63x^9 - 3x^2 - 1)}{x^3(x^2 + 1)^3} = 0$ for $x \approx \pm 1.84$

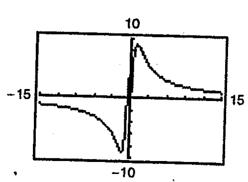
Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$

Minimum: $(-1.10, -9.05)$

Maximum: $(1.10, 9.05)$

Points of inflection: $(-1.84, -7.86)$, $(1.84, 7.86)$



26. $f(x) = x + \dfrac{4}{x^2 + 1} = \dfrac{x^3 + x + 4}{x^2 + 1} = 0$ for $x \approx -1.379$

$f'(x) = \dfrac{x^4 + 2x^2 - 8x + 1}{(x^2 + 1)^2} = 0$ for $x \approx 1.608, x \approx 0.129$

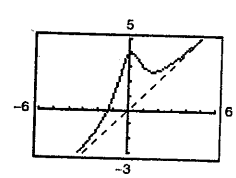
$f''(x) = \dfrac{8(3x^2 - 1)}{(x^2 + 1)^3} = 0$ for $x = \pm\dfrac{1}{\sqrt{3}} \approx \pm 0.577$

Slant asymptote: $y = x$

Points of inflection: $(-0.577, 2.423)$, $(0.577, 3.577)$

Relative maximum: $(0.129, 4.064)$

Relative minimum: $(1.608, 2.724)$



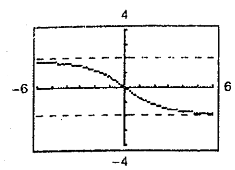
27. $f(x) = \dfrac{-2x}{\sqrt{x^2 + 7}}$

$f'(x) = \dfrac{-14}{\left(x^2 + 7\right)^{3/2}} < 0$

$f''(x) = \dfrac{42x}{\left(x^2 + 7\right)^{5/2}} = 0$ at $x = 0$

Horizontal asymptotes: $y = \pm 2$

Point of inflection: $(0, 0)$



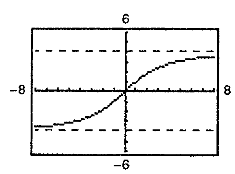
28. $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$

$f'(x) = \dfrac{60}{\left(x^2 + 15\right)^{3/2}} > 0$

$f''(x) = \dfrac{-180x}{\left(x^2 + 15\right)^{5/2}} = 0$ at $x = 0$

Horizontal asymptotes: $y = \pm 4$

Point of inflection: $(0, 0)$



29. $f(x) = 2x - 4\sin x,\ 0 \le x \le 2\pi$

$f'(x) = 2 - 4\cos x$

$f'(x) = 4\sin x$

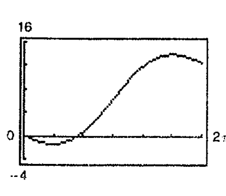
$f'(x) = 0 \Rightarrow \cos x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

$f''(x) = 0 \Rightarrow x = 0, \pi, 2\pi$

Relative minimum: $\left(\dfrac{\pi}{3}, \dfrac{2\pi}{3} - 2\sqrt{3}\right)$

Relative maximum: $\left(\dfrac{5\pi}{3}, \dfrac{10\pi}{3} + 2\sqrt{3}\right)$

Points of inflection: $(0, 0), (\pi, 2\pi), (2\pi, 4\pi)$



30. $f(x) = -x + 2\cos x,\ 0 \le x \le 2\pi$

$f'(x) = -1 - 2\sin x$

$f''(x) = -2\cos x$

$f(x) = 0 \Rightarrow x \approx 1.0299$

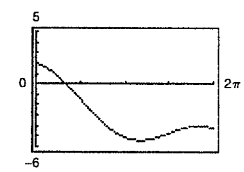
$f'(x) = 0 \Rightarrow \sin x = -\dfrac{1}{2} \Rightarrow x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

$f''(x) = 0 \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Relative minimum: $\left(\dfrac{7\pi}{6}, -\sqrt{3} - \dfrac{7\pi}{6}\right)$

Relative maximum: $\left(\dfrac{11\pi}{6}; \sqrt{3} - \dfrac{11\pi}{6}\right)$

Points of inflection: $\left(\dfrac{\pi}{2}, -\dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, -\dfrac{3\pi}{2}\right)$



31. $y = \cos x - \dfrac{1}{4}\cos 2x,\ 0 \le x \le 2\pi$

$y = 0$ at $x \approx 1.797, 4.486$

$y = -\sin x + \dfrac{1}{2}\sin 2x = -\sin x + \sin x \cos x$

$= \sin x(\cos x - 1)$

$y'' = -\cos x + \cos 2x = -\cos x + 2\cos^2 x - 1$

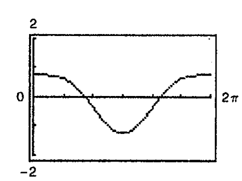
$= (2\cos x + 1)(\cos x - 1)$

$y' = 0 \Rightarrow x = 0, \pi, 2\pi$

$y'' = 0 \Rightarrow x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, 0, 2\pi$

Relative minimum: $\left(\pi, -\dfrac{5}{4}\right)$

Points of inflection: $\left(\dfrac{2\pi}{3}, -\dfrac{3}{8}\right), \left(\dfrac{4\pi}{3}, -\dfrac{3}{8}\right)$



32. $y = 2x - \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$y' = 2 - \sec^2 x = 0$ when $x = \pm\frac{\pi}{4}$.

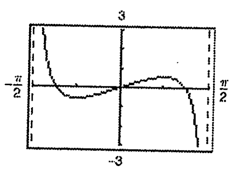
$y'' = -2\sec^2 x \tan x = 0$ when $x = 0$.

Vertical asymptotes: $x = \pm\frac{\pi}{2}$

Relative minimum: $\left(-\frac{\pi}{4}, 1 - \frac{\pi}{2}\right)$

Relative maximum: $\left(\frac{\pi}{4}, \frac{\pi}{2} - 1\right)$

Point of inflection: $(0, 0)$

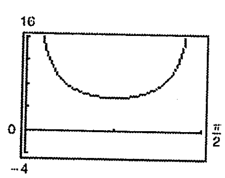


33. $y = 2(\csc x + \sec x), \ 0 < x < \frac{\pi}{2}$

$y' = 2(\sec x \tan x - \csc x \cot x) = 0 \Rightarrow x = \frac{\pi}{4}$

Relative minimum: $\left(\frac{\pi}{4}, 4\sqrt{2}\right)$

Vertical asymptotes: $x = 0, \frac{\pi}{2}$



34. $g(x) = x\cot x, \quad -2\pi < x < 2\pi$

$g'(x) = \frac{\sin x \cos x - x}{\sin^2 x}$

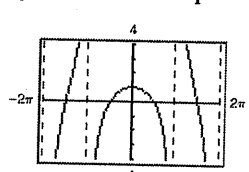
$g(0)$ does not exist, but $\lim_{x\to\infty} x\cot x = 1$.

$g''(x) = \frac{2(x\cos x - \sin x)}{\sin^3 x}$

Vertical asymptotes: $x = \pm 2\pi, \pm\pi$

Intercepts: $\left(\pm\frac{3\pi}{2}, 0\right), \left(\pm\frac{\pi}{2}, 0\right)$

Symmetric with respect to y-axis



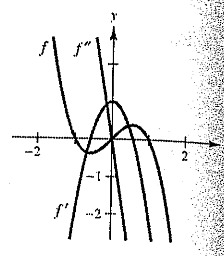
35. Because the slope is negative, the function is decreasing on $(2, 8)$, and so $f(3) > f(5)$.

36. If $f'(x) = 2$ in $[-5, 5]$, then $f(x) = 2x + 3$ and $f(2) = 7$ is the least possible value of $f(2)$. If $f'(x) = 4$ in $[-5, 5]$, then $f(x) = 4x + 3$ and $f(2) = 11$ is the greatest possible value of $f(2)$.

37. f is cubic.

f' is quadratic.

f'' is linear.

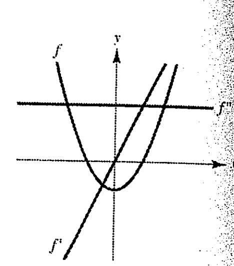


The zeros of f' correspond to the points where the graph of f has horizontal tangents. The zero of f'' corresponds to the point where the graph of f' has a horizontal tangent.

38. f'' is constant.

f' is linear.

f is quadratic.

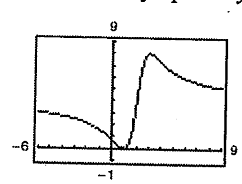


The zero of f' corresponds to the points where the graph of f has a horizontal tangent. There are no zeros on of f'', which means the graph of f' has no horizontal tangent.

39. $f(x) = \frac{4(x - 1)^2}{x^2 - 4x + 5}$

Vertical asymptote: none

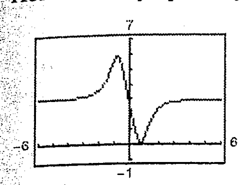
Horizontal asymptote: $y = 4$



The graph crosses the horizontal asymptote $y = 4$. If a function has a vertical asymptote at $x = c$, the graph would not cross it because $f(c)$ is undefined.

40. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

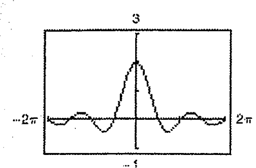
Vertical asymptote: none
Horizontal asymptote: $y = 3$



The graph crosses the horizontal asymptote $y = 3$. If a function has a vertical asymptote at $x = c$, the graph would not cross it because $f(c)$ is undefined.

41. $h(x) = \dfrac{\sin 2x}{x}$

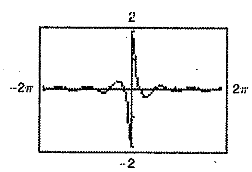
Vertical asymptote: none
Horizontal asymptote: $y = 0$



Yes, it is possible for a graph to cross its horizontal asymptote.
It is not possible to cross a vertical asymptote because the function is not continuous there.

42. $f(x) = \dfrac{\cos 3x}{4x}$

Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 0$

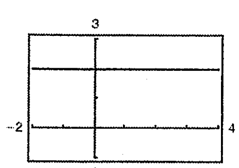


Yes, it is possible for a graph to cross its horizontal asymptote.
It is not possible to cross a vertical asymptote because the function is not continuous there.

43. $h(x) = \dfrac{6 - 2x}{3 - x}$

$$= \frac{2(3 - x)}{3 - x} = \begin{cases} 2, & \text{if } x \neq 3 \\ \text{Undefined}, & \text{if } x = 3 \end{cases}$$

The rational function is not reduced to lowest terms.

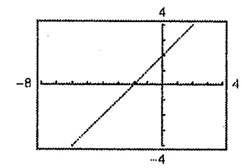


There is a hole at $(3, 2)$.

44. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

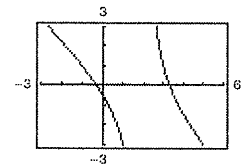
$$= \frac{(x + 2)(x - 1)}{x - 1} = \begin{cases} x + 2, & \text{if } x \neq 1 \\ \text{Undefined}, & \text{if } x = 1 \end{cases}$$

The rational function is not reduced to lowest terms.



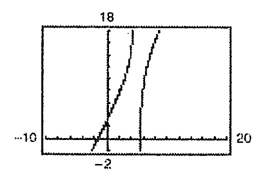
There is a hole at $(1, 3)$.

45. $f(x) = -\dfrac{x^2 - 3x - 1}{x - 2} = -x + 1 + \dfrac{3}{x - 2}$



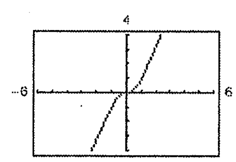
The graph appears to approach the slant asymptote $y = -x + 1$.

46. $g(x) = \dfrac{2x^2 - 8x - 15}{x - 5} = 2x + 2 - \dfrac{5}{x - 5}$



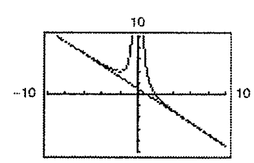
The graph appears to approach the slant asymptote $y = 2x + 2$.

47. $f(x) = \dfrac{2x^3}{x^2 + 1} = 2x - \dfrac{2x}{x^2 + 1}$



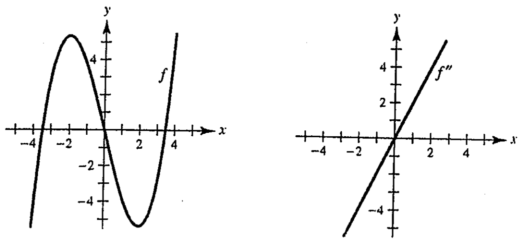
The graph appears to approach the slant asymptote $y = 2x$.

48. $h(x) = \dfrac{-x^3 + x^2 + 4}{x^2} = -x + 1 + \dfrac{4}{x^2}$



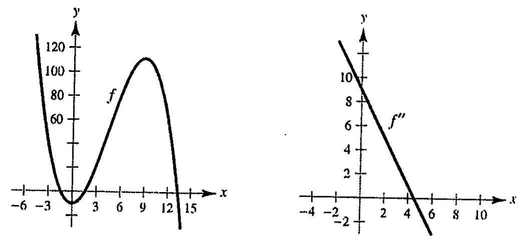
The graph appears to approach the slant asymptote $y = -x + 1$.

49.



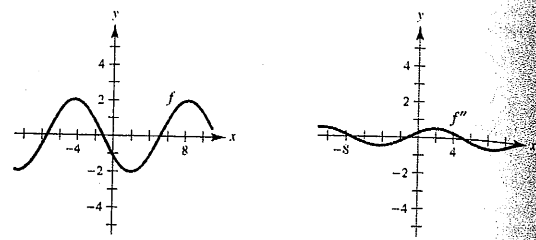
(or any vertical translation of f)

50.



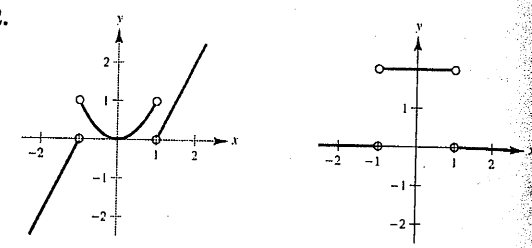
(or any vertical translation of f)

51.



(or any vertical translation of f)

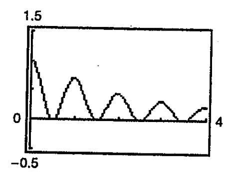
52.



(or any vertical translation of the 3 segments of f)

53. $f(x) = \dfrac{\cos^2 \pi x}{\sqrt{x^2 + 1}}, (0, 4)$

(a)



On $(0, 4)$ there seem to be 7 critical numbers: 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5

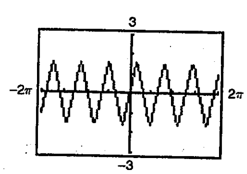
(b) $f'(x) = \dfrac{-\cos \pi x\left(x \cos \pi x + 2\pi\left(x^2 + 1\right)\sin \pi x\right)}{\left(x^2 + 1\right)^{3/2}} = 0$

Critical numbers $\approx \frac{1}{2}, 0.97, \frac{3}{2}, 1.98, \frac{5}{2}, 2.98, \frac{7}{2}$.

The critical numbers where maxima occur appear to be integers in part (a), but approximating them using f' shows that they are not integers.

54. $f(x) = \tan(\sin \pi x)$

(a)



(b) $f(-x) = \tan(\sin(-\pi x)) = \tan(-\sin \pi x)$

$= -\tan(\sin \pi x) = -f(x)$

Symmetry with respect to the origin

(c) Periodic with period 2

(d) On $(-1, 1)$, there is a relative maximum at $\left(\frac{1}{2}, \tan 1\right)$ and a relative minimum at $\left(-\frac{1}{2}, -\tan 1\right)$.

(e) On $(0, 1)$, the graph of f is concave downward.

55. Vertical asymptote: $x = 3$

Horizontal asymptote: $y = 0$

$$y = \frac{1}{x - 3}$$

56. Vertical asymptote: $x = -5$

Horizontal asymptote: none

$$y = \frac{x^2}{x + 5}$$

57. Vertical asymptote: $x = 3$

Slant asymptote: $y = 3x + 2$

$$y = 3x + 2 + \frac{1}{x - 3} = \frac{3x^2 - 7x - 5}{x - 3}$$

58. Vertical asymptote: $x = 2$

Slant asymptote: $y = -x$

$$y = -x + \frac{1}{x - 2} = \frac{-x^2 + 2x + 1}{x - 2}$$

59. (a) $f'(x) = 0$ at x_0, x_2 and x_4 (horizontal tangent).

(b) $f''(x) = 0$ at x_2 and x_3 (point of inflection).

(c) $f'(x)$ does not exist at x_1 (sharp corner).

(d) f has a relative maximum at x_1.

(e) f has a point of inflection at x_2 and x_3 (change in concavity).

60. (a) $f'(x) = 0$ for $x = -2$ (relative maximum) and $x = 2$ (relative minimum).

f' is negative for $-2 < x < 2$ (decreasing).

f' is positive for $x > 2$ and $x < -2$ (increasing).

(b) $f''(x) = 0$ at $x = 0$ (point of inflection).

f'' is positive for $x > 0$ (concave upward).

f'' is negative for $x < 0$ (concave downward).

(c) f' is increasing on $(0, \infty)$. $(f'' > 0)$

(d) $f'(x)$ is minimum at $x = 0$. The rate of change of f at $x = 0$ is less than the rate of change of f for all other values of x.

61. Tangent line at P: $y - y_0 = f'(x_0)(x - x_0)$

(a) Let $y = 0$: $-y_0 = f'(x_0)(x - x_0)$

$$f'(x_0)x = x_0 f'(x_0) - y_0$$

$$x = x_0 - \frac{y_0}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

x-intercept: $\left(x_0 - \frac{f(x_0)}{f'(x_0)}, 0\right)$

(b) Let $x = 0$: $y - y_0 = f'(x_0)(-x_0)$

$$y = y_0 - x_0 f'(x_0)$$

$$y = f(x_0) - x_0 f'(x_0)$$

y-intercept: $\left(0, f(x_0) - x_0 f'(x_0)\right)$

(c) Normal line: $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

Let $y = 0$: $-y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

$$-y_0 f'(x_0) = -x + x_0$$

$$x = x_0 + y_0 f'(x_0) = x_0 + f(x_0)f'(x_0)$$

x-intercept: $\left(x_0 + f(x_0)f'(x_0), 0\right)$

(d) Let $x = 0$: $y - y_0 = \frac{-1}{f'(x_0)}(-x_0)$

$$y = y_0 + \frac{x_0}{f'(x_0)}$$

y-intercept: $\left(0, y_0 + \frac{x_0}{f'(x_0)}\right)$

(e) $|BC| = \left|x_0 - \frac{f(x_0)}{f'(x_0)} - x_0\right| = \left|\frac{f(x_0)}{f'(x_0)}\right|$

(f) $|PC|^2 = y_0^2 + \left(\frac{f(x_0)}{f'(x_0)}\right) = \frac{f(x_0)^2 f'(x_0)^2 + f(x_0)^2}{f'(x_0)^2}$

$$|PC| = \left|\frac{f(x_0)\sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|$$

(g) $|AB| = \left|x_0 - \left(x_0 + f(x_0)f'(x_0)\right)\right| = \left|f(x_0)f'(x_0)\right|$

(h) $|AP|^2 = f(x_0)^2 f'(x_0)^2 + y_0^2$

$$|AP| = |f(x_0)|\sqrt{1 + [f'(x_0)]^2}$$

62. $f(x) = \frac{2x^n}{x^4 + 1}$

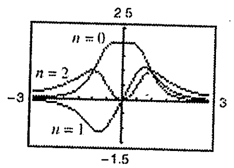
(a) For n even, f is symmetric about the y-axis. For n odd, f is symmetric about the origin.

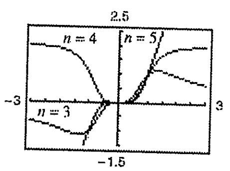
(b) The x-axis will be the horizontal asymptote if the degree of the numerator is less than 4. That is, $n = 0, 1, 2, 3$.

(c) $n = 4$ gives $y = 2$ as the horizontal asymptote.

(d) There is a slant asymptote $y = 2x$ if $n = 5$: $\frac{2x^5}{x^4 + 1} = 2x - \frac{2x}{x^4 + 1}$.

(e)





n	0	1	2	3	4	5
M	1	2	3	2	1	0
N	2	3	4	5	2	3

63. $f(x) = \frac{ax}{(x - b)^2}$

Answers will vary. *Sample answer*: The graph has a vertical asymptote at $x = b$. If a and b are both positive, or both negative, then the graph of f approaches ∞ as x approaches b, and the graph has a minimum at $x = -b$. If a and b have opposite signs, then the graph of f approaches $-\infty$ as x approaches b, and the graph has a maximum at $x = -b$.

64. $f(x) = \frac{1}{2}(ax)^2 - (ax) = \frac{1}{2}(ax)(ax - 2), a \neq 0$

$f'(x) = a^2x - a = a(ax - 1) = 0$ when $x = \frac{1}{a}$.

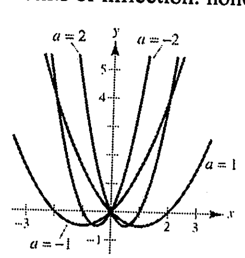
$f''(x) = a^2 > 0$ for all x.

(a) Intercepts: $(0, 0), \left(\frac{2}{a}, 0\right)$

Relative minimum: $\left(\frac{1}{a}, -\frac{1}{2}\right)$

Points of inflection: none

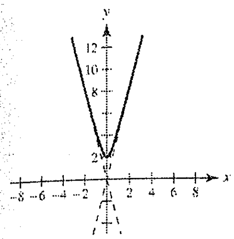
(b)



65. $y = \sqrt{4 + 16x^2}$

As $x \to \infty, y \to 4x$. As $x \to -\infty, y \to -4x$.

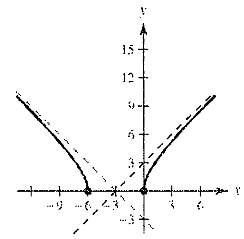
Slant asymptotes: $y = \pm 4x$



66. $y = \sqrt{x^2 + 6x} = \sqrt{(x + 3)^2 - 9}$

$y \to x + 3$ as $x \to \infty$, and $y \to -x - 3$ as $x \to -\infty$.

Slant asymptotes: $y = x + 3, y = -x - 3$



67. Let $\lambda = \dfrac{\dfrac{f(x) - f(a)}{x - a} - \dfrac{f(b) - f(a)}{b - a}}{x - b}, a < x < b$.

$$\lambda(x - b) = \frac{f(x) - f(a)}{x - a} - \frac{f(b) - f(a)}{b - a}$$

$$\lambda(x - b)(x - a) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

$$f(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) + \lambda(x - b)(x - a)$$

Let $h(t) = f(t) - \left\{f(a) + \dfrac{f(b) - f(a)}{b - a}(t - a) + \lambda(t - a)(t - b)\right\}$.

$h(a) = 0, h(b) = 0, h(x) = 0$

By Rolle's Theorem, there exist numbers α_1 and α_2 such that $a < \alpha_1 < x < \alpha_2 < b$ and $h'(\alpha_1) = h'(\alpha_2) = 0$.

By Rolle's Theorem, there exists β in (a, b) such that $h''(\beta) = 0$.

Finally,

$0 = h''(\beta) = f''(\beta) - \{2\lambda\} \Rightarrow \lambda = \frac{1}{2}f''(\beta)$.