## Section 3.7 Optimization Problems

**1.** (a)

| First Number, $x$ | Second Number | Product, $P$ |
|---|---|---|
| 10 | $110 - 10$ | $10(110 - 10) = 1000$ |
| 20 | $110 - 20$ | $20(110 - 20) = 1800$ |
| 30 | $110 - 30$ | $30(110 - 30) = 2400$ |
| 40 | $110 - 40$ | $40(110 - 40) = 2800$ |
| 50 | $110 - 50$ | $50(110 - 50) = 3000$ |
| 60 | $110 - 60$ | $60(110 - 60) = 3000$ |

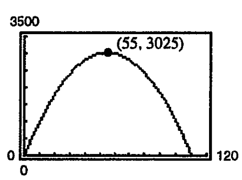
(b)

| First Number, $x$ | Second Number | Product, $P$ |
|---|---|---|
| 10 | $110 - 10$ | $10(110 - 10) = 1000$ |
| 20 | $110 - 20$ | $20(110 - 20) = 1800$ |
| 30 | $110 - 30$ | $30(110 - 30) = 2400$ |
| 40 | $110 - 40$ | $40(110 - 40) = 2800$ |
| 50 | $110 - 50$ | $50(110 - 50) = 3000$ |
| 60 | $110 - 60$ | $60(110 - 60) = 3000$ |
| 70 | $110 - 70$ | $70(110 - 70) = 2800$ |
| 80 | $110 - 80$ | $80(110 - 80) = 2400$ |
| 90 | $110 - 90$ | $90(110 - 90) = 1800$ |
| 100 | $110 - 100$ | $100(110 - 100) = 1000$ |

The maximum is attained near $x = 50$ and 60.

(c) $P = x(110 - x) = 110x - x^2$

(d)




The solution appears to be $x = 55$.

(e) $\frac{dP}{dx} = 110 - 2x = 0$ when $x = 55$.

$\frac{d^2P}{dx^2} = -2 < 0$

$P$ is a maximum when $x = 110 - x = 55$. The two numbers are 55 and 55.

2. (a)

| Height, $x$ | Length & Width | Volume |
|---|---|---|
| 1 | $24-2(1)$ | $1[24-2(1)]^2 = 484$ |
| 2 | $24-2(2)$ | $2[24-2(2)]^2 = 800$ |
| 3 | $24-2(3)$ | $3[24-2(3)]^2 = 972$ |
| 4 | $24-2(4)$ | $4[24-2(4)]^2 = 1024$ |
| 5 | $24-2(5)$ | $5[24-2(5)]^2 = 980$ |
| 6 | $24-2(6)$ | $6[24-2(6)]^2 = 864$ |

The maximum is attained near $x = 4$.

(b) $V = x(24 - 2x)^2, 0 < x < 12$

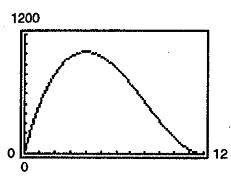
(c) $\frac{dV}{dx} = 2x(24 - 2x)(-2) + (24 - 2x)^2 = (24 - 2x)(24 - 6x)$

$= 12(12 - x)(4 - x) = 0$ when $x = 12, 4$ (12 is not in the domain).

$\frac{d^2V}{dx^2} = 12(2x - 16)$

$\frac{d^2V}{dx^2} < 0$ when $x = 4$.

When $x = 4, V = 1024$ is maximum.

(d) 


The maximum volume seems to be 1024.

3. Let $x$ and $y$ be two positive numbers such that $x + y = S$.

$P = xy = x(S - x) = Sx - x^2$

$\frac{dP}{dx} = S - 2x = 0$ when $x = \frac{S}{2}$.

$\frac{d^2P}{dx^2} = -2 < 0$ when $x = \frac{S}{2}$.

$P$ is a maximum when $x = y = S/2$.

4. Let $x$ and $y$ be two positive numbers such that $xy = 185$.

$S = x + y = x + \frac{185}{x}$

$\frac{dS}{dx} = 1 - \frac{185}{x^2} = 0$ when $x = \sqrt{185}$.

$\frac{d^2S}{dx^2} = \frac{370}{x^3} > 0$ when $x = \sqrt{185}$

$S$ is a minimum when $x = y = \sqrt{185}$.

5. Let $x$ and $y$ be two positive numbers such that $xy = 147$.

$S = x + 3y = \frac{147}{y} + 3y$

$\frac{dS}{dy} = 3 - \frac{147}{y^2} = 0$ when $y = 7$.

$\frac{d^2S}{dy^2} = \frac{294}{y^3} > 0$ when $y = 7$.

$S$ is minimum when $y = 7$ and $x = 21$.

6. Let $x$ be a positive number.

$S = x + \frac{1}{x}$

$\frac{dS}{dx} = 1 - \frac{1}{x^2} = 0$ when $x = 1$.

$\frac{d^2S}{dx^2} = \frac{2}{x^3} > 0$ when $x = 1$.

The sum is a minimum when $x = 1$ and $1/x = 1$.

**7.** Let $x$ and $y$ be two positive numbers such that $x + 2y = 108$.

$$P = xy = y(108 - 2y) = 108y - 2y^2$$

$$\frac{dP}{dy} = 108 - 4y = 0 \text{ when } y = 27.$$

$$\frac{d^2P}{dy^2} = -4 < 0 \text{ when } y = 27.$$

$P$ is a maximum when $x = 54$ and $y = 27$.

**8.** Let $x$ and $y$ be two positive numbers such that $x^2 + y = 54$.

$$P = xy = x(54 - x^2) = 54x - x^3$$

$$\frac{dP}{dx} = 54 - 3x^2 = 0 \text{ when } x = 3\sqrt{2}.$$

$$\frac{d^2P}{dx^2} = -6x < 0 \text{ when } x = 3\sqrt{2}.$$

The product is a maximum when $x = 3\sqrt{2}$ and $y = 36$.

**9.** Let $x$ be the length and $y$ the width of the rectangle.

$$2x + 2y = 80$$

$$y = 40 - x$$

$$A = xy = x(40 - x) = 40x - x^2$$

$$\frac{dA}{dx} = 40 - 2x = 0 \text{ when } x = 20.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = 20.$$

$A$ is maximum when $x = y = 20$ m.

**10.** Let $x$ be the length and $y$ the width of the rectangle.

$$2x + 2y = P$$

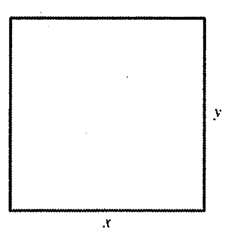
$$y = \frac{P - 2x}{2} = \frac{P}{2} - x$$

$$A = xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$$

$$\frac{dA}{dx} = \frac{P}{2} - 2x = 0 \text{ when } x = \frac{P}{4}.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = \frac{P}{4}.$$

$A$ is maximum when $x = y = P/4$ units. (A square!)




**11.** Let $x$ be the length and $y$ the width of the rectangle.

$$xy = 32$$

$$y = \frac{32}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{32}{x}\right) = 2x + \frac{64}{x}$$

$$\frac{dP}{dx} = 2 - \frac{64}{x^2} = 0 \text{ when } x = 4\sqrt{2}.$$

$$\frac{d^2P}{dx^2} = \frac{128}{x^3} > 0 \text{ when } x = 4\sqrt{2}.$$

$P$ is minimum when $x = y = 4\sqrt{2}$ ft.

**12.** Let $x$ be the length and $y$ the width of the rectangle.

$$xy = A$$

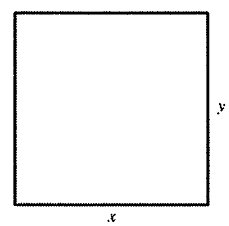
$$y = \frac{A}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{A}{x}\right) = 2x + \frac{2A}{x}$$

$$\frac{dP}{dx} = 2 - \frac{2A}{x^2} = 0 \text{ when } x = \sqrt{A}.$$

$$\frac{d^2P}{dx^2} = \frac{4A}{x^3} > 0 \text{ when } x = \sqrt{A}.$$

$P$ is minimum when $x = y = \sqrt{A}$ cm. (A square!)




**13.** $d = \sqrt{(x - 2)^2 + \left[x^2 - (1/2)\right]^2}$

$= \sqrt{x^4 - 4x + (17/4)}$

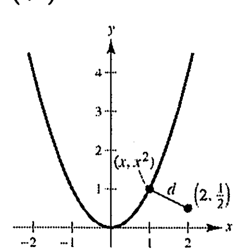
Because $d$ is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^4 - 4x + \tfrac{17}{4}.$$

$$f'(x) = 4x^3 - 4 = 0$$

$$x = 1$$

By the First Derivative Test, the point nearest to $\left(2, \frac{1}{2}\right)$ is $(1, 1)$.



**14.** $f(x) = (x-1)^2, (-5, 3)$

$$d = \sqrt{(x+5)^2 + \left[(x-1)^2 - 3\right]^2}$$
$$= \sqrt{(x^2+10x+25) + (x^2-2x-2)^2}$$
$$= \sqrt{(x^2+10x+25) + (x^4-4x^3+8x+4)}$$
$$= \sqrt{x^4-4x^3+x^2+18x+29}$$

Because $d$ is smallest when the expression inside the radical is smallest, you need to find the critical numbers of

$$g(x) = x^4 - 4x^3 + x^2 + 18x + 29$$
$$g'(x) = 4x^3 - 12x^2 + 2x + 18$$
$$= 2(x+1)(2x^2 - 8x + 9) = 0$$
$$x = -1$$

By the First Derivative Test, $x = -1$ yields a minimum. So, $(-1, 4)$ is closest to $(-5, 3)$.

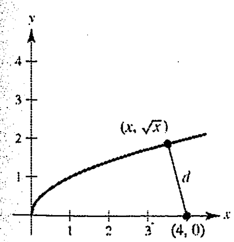
**15.** $d = \sqrt{(x-4)^2 + (\sqrt{x} - 0)^2}$

$$= \sqrt{x^2 - 7x + 16}$$

Because $d$ is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^2 - 7x + 16.$$
$$f'(x) = 2x - 7 = 0$$
$$x = \tfrac{7}{2}$$

By the First Derivative Test, the point nearest to $(4, 0)$ is $\left(7/2, \sqrt{7/2}\right)$.




**16.** $f(x) = \sqrt{x-8}, (12, 0)$

$$d = \sqrt{(x-12)^2 + \left(\sqrt{x-8} - 0\right)^2}$$
$$= \sqrt{x^2 - 24x + 144 + x - 8}$$
$$= \sqrt{x^2 - 23x + 136}$$

Because $d$ is smallest when the expression inside the radical is smallest, you need to find the critical numbers of

$$g(x) = x^2 - 23x + 136$$
$$g'(x) = 2x - 23 = 0 \text{ when } x = \frac{23}{2}$$
$$g''(x) = 2 > 0 \text{ at } x = \frac{23}{2}$$

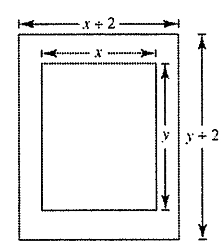
The point nearest to $(12, 0)$ is

$$\left(\frac{23}{2}, f\left(\frac{23}{2}\right)\right) = \left(\frac{23}{2}, \frac{\sqrt{14}}{2}\right)$$

**17.** $xy = 30 \Rightarrow y = \dfrac{30}{x}$

$$A = (x+2)\left(\frac{30}{x} + 2\right) \text{ (see figure)}$$
$$\frac{dA}{dx} = (x+2)\left(\frac{-30}{x^2}\right) + \left(\frac{30}{x} + 2\right)$$
$$= \frac{2(x^2 - 30)}{x^2} = 0 \text{ when } x = \sqrt{30}.$$
$$y = \frac{30}{\sqrt{30}} = \sqrt{30}$$

By the First Derivative Test, the dimensions $(x+2)$ by $(y+2)$ are $(2+\sqrt{30})$ by $(2+\sqrt{30})$ (approximately 7.477 by 7.477). These dimensions yield a minimum area.



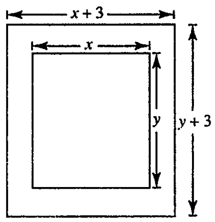
**18.** $xy = 36 \Rightarrow y = \dfrac{36}{x}$

$$A = (x+3)(y+3) = (x+3)\left(\frac{36}{x} + 3\right)$$

$$= 36 + \frac{108}{x} + 3x + 9$$

$$\frac{dA}{dx} = \frac{-108}{x^2} + 3 = 0 \Rightarrow 3x^2 = 108 \Rightarrow x = 6, y = 6$$

Dimensions: $9 \times 9$




**19.** $xy = 245{,}000$ (see figure)

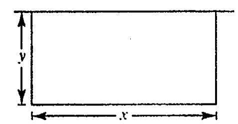
$$S = x + 2y$$

$$= \left(x + \frac{490{,}000}{x}\right) \text{ where } S \text{ is the length of fence needed.}$$

$$\frac{dS}{dx} = 1 - \frac{490{,}000}{x^2} = 0 \text{ when } x = 700.$$

$$\frac{d^2S}{dx^2} = \frac{980{,}000}{x^3} > 0 \text{ when } x = 700.$$

$S$ is a minimum when $x = 700$ m and $y = 350$ m.




**20.** $S = 2x^2 + 4xy = 337.5$

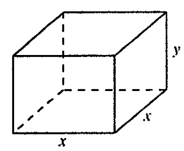
$$y = \frac{337.5 - 2x^2}{4x}$$

$$V = x^2y = x^2\left[\frac{337.5 - 2x^2}{4x}\right] = 84.375x - \frac{1}{2}x^3$$

$$\frac{dV}{dx} = 84.375 - \frac{3}{2}x^2 = 0 \Rightarrow x^2 = 56.25 \Rightarrow x = 7.5 \text{ and } y = 7.5.$$

$$\frac{d^2V}{dx^2} = -3x < 0 \text{ for } x = 7.5.$$

The maximum value occurs when $x = y = 7.5$ cm.




**21.** $16 = 2y + x + \pi\left(\dfrac{x}{2}\right)$

$$32 = 4y + 2x + \pi x$$

$$y = \frac{32 - 2x - \pi x}{4}$$

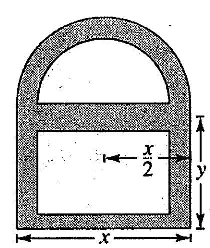
$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \left(\frac{32 - 2x - \pi x}{4}\right)x + \frac{\pi x^2}{8} = 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$

$$\frac{dA}{dx} = 8 - x - \frac{\pi}{2}x + \frac{\pi}{4}x = 8 - x\left(1 + \frac{\pi}{4}\right) = 0 \text{ when } x = \frac{8}{1 + (\pi/4)} = \frac{32}{4 + \pi}.$$

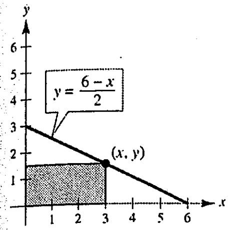
$$\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0 \text{ when } x = \frac{32}{4 + \pi}.$$

$$y = \frac{32 - 2[32/(4+\pi)] - \pi[32/(4+\pi)]}{4} = \frac{16}{4 + \pi}$$

The area is maximum when $y = \dfrac{16}{4 + \pi}$ ft and $x = \dfrac{32}{4 + \pi}$ ft.



22. You can see from the figure that $A = xy$ and $y = \dfrac{6 - x}{2}$.




$$A = x\left(\frac{6 - x}{2}\right) = \frac{1}{2}\left(6x - x^2\right).$$

$$\frac{dA}{dx} = \frac{1}{2}(6 - 2x) = 0 \text{ when } x = 3.$$

$$\frac{d^2A}{dx^2} = -1 < 0 \text{ when } x = 3.$$

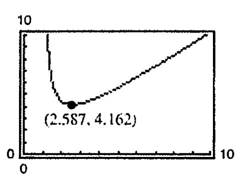
$A$ is a maximum when $x = 3$ and $y = 3/2$.

23. (a) $\dfrac{y - 2}{0 - 1} = \dfrac{0 - 2}{x - 1}$

$$y = 2 + \frac{2}{x - 1}$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(2 + \frac{2}{x - 1}\right)^2} = \sqrt{x^2 + 4 + \frac{8}{x - 1} + \frac{4}{(x - 1)^2}}, \quad x > 1$$

(b)




$L$ is minimum when $x \approx 2.587$ and $L \approx 4.162$.

(c) Area $= A(x) = \dfrac{1}{2}xy = \dfrac{1}{2}x\left(2 + \dfrac{2}{x - 1}\right) = x + \dfrac{x}{x - 1}$

$$A'(x) = 1 + \frac{(x - 1) - x}{(x - 1)^2} = 1 - \frac{1}{(x - 1)^2} = 0$$

$$(x - 1)^2 = 1$$

$$x - 1 = \pm 1$$

$$x = 0, 2 \text{ (select } x = 2)$$

They $y = 4$ and $A = 4$.

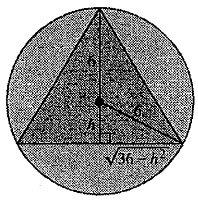
Vertices: $(0, 0)$, $(2, 0)$, $(0, 4)$

**24.** (a) $A = \frac{1}{2}\text{ base} \times \text{height} = \frac{1}{2}\left(2\sqrt{36 - h^2}\right)(6 + h) = \sqrt{36 - h^2}(6 + h)$

$$\frac{dA}{dh} = \frac{1}{2}\left(36 - h^2\right)^{-1/2}(-2h)(6 + h) + \left(36 - h^2\right)^{1/2}$$

$$= \left(36 - h^2\right)^{-1/2}\left[-h(6 + h) + \left(36 - h^2\right)\right] = \frac{-2\left(h^2 + 3h - 18\right)}{\sqrt{36 - h^2}} = \frac{-2(h + 6)(h - 3)}{\sqrt{36 - h^2}}$$

$\frac{dA}{dh} = 0$ when $h = 3$, which is a maximum by the First Derivative Test. So, the sides are $2\sqrt{36 - h^2} = 6\sqrt{3}$, an equilateral triangle. Area $= 27\sqrt{3}$ sq. units.




(b)

$$\cos\alpha = \frac{6 + h}{2\sqrt{3}\sqrt{6 + h}} = \frac{\sqrt{6 + h}}{2\sqrt{3}}$$

$$\tan\alpha = \frac{\sqrt{36 - h^2}}{6 + h}$$

$$\text{Area} = 2\left(\frac{1}{2}\right)\left(\sqrt{36 - h^2}\right)(6 + h) = (6 + h)^2\tan\alpha = 144\cos^4\alpha\tan\alpha$$

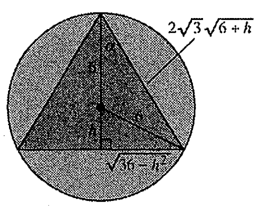
$$A'(\alpha) = 144\left[\cos^4\alpha\sec^2\alpha + 4\cos^3(-\sin\alpha)\tan\alpha\right] = 0$$

$$\Rightarrow \cos^4\alpha\sec^2\alpha = 4\cos^3\alpha\sin\alpha\tan\alpha$$

$$1 = 4\cos\alpha\sin\alpha\tan\alpha$$

$$\frac{1}{4} = \sin^2\alpha$$

$$\sin\alpha = \frac{1}{2} \Rightarrow \alpha = 30° \text{ and } A = 27\sqrt{3}.$$




(c) Equilateral triangle

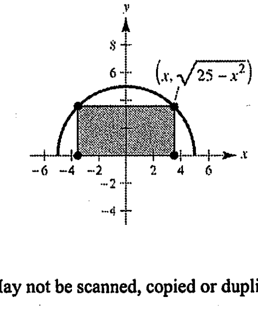
**25.** $A = 2xy = 2x\sqrt{25 - x^2}$ (see figure)

$$\frac{dA}{dx} = 2x\left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{25 - x^2}}\right) + 2\sqrt{25 - x^2} = 2\left(\frac{25 - 2x^2}{\sqrt{25 - x^2}}\right) = 0 \text{ when } x = y = \frac{5\sqrt{2}}{2} \approx 3.54.$$

By the First Derivative Test, the inscribed rectangle of maximum area has vertices

$$\left(\pm\frac{5\sqrt{2}}{2}, 0\right), \left(\pm\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right).$$

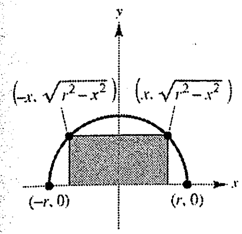
Width: $\frac{5\sqrt{2}}{2}$; Length: $5\sqrt{2}$



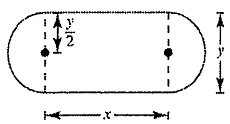
**26.** $A = 2xy = 2x\sqrt{r^2 - x^2}$ (see figure)

$$\frac{dA}{dx} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} = 0 \text{ when } x = \frac{\sqrt{2}r}{2}.$$

By the First Derivative Test, $A$ is maximum when the rectangle has dimensions $\sqrt{2}r$ by $(\sqrt{2}r)/2$.




**27.** (a)
$$\begin{aligned} P &= 2x + 2\pi r \\ &= 2x + 2\pi\left(\frac{y}{2}\right) \\ &= 2x + \pi y = 200 \\ \Rightarrow y &= \frac{200 - 2x}{\pi} = \frac{2}{\pi}(100 - x) \end{aligned}$$




(b)

| Length, $x$ | Width, $y$ | Area, $xy$ |
|---|---|---|
| 10 | $\frac{2}{\pi}(100 - 10)$ | $(10)\frac{2}{\pi}(100 - 10) \approx 573$ |
| 20 | $\frac{2}{\pi}(100 - 20)$ | $(20)\frac{2}{\pi}(100 - 20) \approx 1019$ |
| 30 | $\frac{2}{\pi}(100 - 30)$ | $(30)\frac{2}{\pi}(100 - 30) \approx 1337$ |
| 40 | $\frac{2}{\pi}(100 - 40)$ | $(40)\frac{2}{\pi}(100 - 40) \approx 1528$ |
| 50 | $\frac{2}{\pi}(100 - 50)$ | $(50)\frac{2}{\pi}(100 - 50) \approx 1592$ |
| 60 | $\frac{2}{\pi}(100 - 60)$ | $(60)\frac{2}{\pi}(100 - 60) \approx 1528$ |

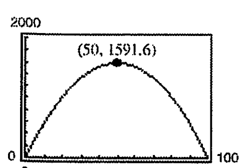
The maximum area of the rectangle is approximately 1592 $m^2$.

(c) $A = xy = x\frac{2}{\pi}(100 - x) = \frac{2}{\pi}(100x - x^2)$

(d) $A' = \frac{2}{\pi}(100 - 2x)$. $A' = 0$ when $x = 50$.

Maximum value is approximately 1592 when length = 50 m and width = $\frac{100}{\pi}$.

(e)




Maximum area is approximately 1591.55 $m^2$ ($x = 50$ m).

**28.** $V = \pi r^2 h = 22$ cubic inches or $h = \dfrac{22}{\pi r^2}$

(a)

| Radius, $r$ | Height | Surface Area |
|---|---|---|
| 0.2 | $\dfrac{22}{\pi(0.2)^2}$ | $2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$ |
| 0.4 | $\dfrac{22}{\pi(0.4)^2}$ | $2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.0$ |
| 0.6 | $\dfrac{22}{\pi(0.6)^2}$ | $2\pi(0.6)\left[0.6 + \dfrac{22}{\pi(0.6)^2}\right] \approx 75.6$ |
| 0.8 | $\dfrac{22}{\pi(0.8)^2}$ | $2\pi(0.8)\left[0.8 + \dfrac{22}{\pi(0.8)^2}\right] \approx 59.0$ |

(b)

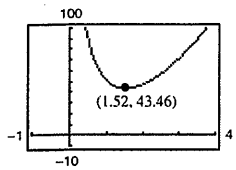
| Radius, $r$ | Height | Surface Area |
|---|---|---|
| 0.2 | $\dfrac{22}{\pi(0.2)^2}$ | $2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$ |
| 0.4 | $\dfrac{22}{\pi(0.4)^2}$ | $2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.0$ |
| 0.6 | $\dfrac{22}{\pi(0.6)^2}$ | $2\pi(0.6)\left[0.6 + \dfrac{22}{\pi(0.6)^2}\right] \approx 75.6$ |
| 0.8 | $\dfrac{22}{\pi(0.8)^2}$ | $2\pi(0.8)\left[0.8 + \dfrac{22}{\pi(0.8)^2}\right] \approx 59.0$ |
| 1.0 | $\dfrac{22}{\pi(1.0)^2}$ | $2\pi(1.0)\left[1.0 + \dfrac{22}{\pi(1.0)^2}\right] \approx 50.3$ |
| 1.2 | $\dfrac{22}{\pi(1.2)^2}$ | $2\pi(1.2)\left[1.2 + \dfrac{22}{\pi(1.2)^2}\right] \approx 45.7$ |
| 1.4 | $\dfrac{22}{\pi(1.4)^2}$ | $2\pi(1.4)\left[1.4 + \dfrac{22}{\pi(1.4)^2}\right] \approx 43.7$ |
| 1.6 | $\dfrac{22}{\pi(1.6)^2}$ | $2\pi(1.6)\left[1.6 + \dfrac{22}{\pi(1.6)^2}\right] \approx 43.6$ |
| 1.8 | $\dfrac{22}{\pi(1.8)^2}$ | $2\pi(1.8)\left[1.8 + \dfrac{22}{\pi(1.8)^2}\right] \approx 44.8$ |
| 2.0 | $\dfrac{22}{\pi(2.0)^2}$ | $2\pi(2.0)\left[2.0 + \dfrac{22}{\pi(2.0)^2}\right] \approx 47.1$ |

The minimum seems to the about 43.6 for $r = 1.6$.

(c) $S = 2\pi r^2 + 2\pi rh$

$$= 2\pi r(r + h) = 2\pi r\left[r + \frac{22}{\pi r^2}\right] = 2\pi r^2 + \frac{44}{r}$$

(d)




The minimum seems to be 43.46 for $r \approx 1.52$.

(e) $\frac{dS}{dr} = 4\pi r - \frac{44}{r^2} = 0$ when $r = \sqrt[3]{11/\pi} \approx 1.52$ in.

$h = \frac{22}{\pi r^2} \approx 3.04$ in.

**Note:** Notice that $h = \frac{22}{\pi r^2} = \frac{22}{\pi(11/\pi)^{2/3}} = 2\left(\frac{11^{1/3}}{\pi^{1/3}}\right) = 2r$.

**29.** Let $x$ be the sides of the square ends and $y$ the length of the package.

$P = 4x + y = 108 \Rightarrow y = 108 - 4x$

$V = x^2y = x^2(108 - 4x) = 108x^2 - 4x^3$

$\frac{dV}{dx} = 216x - 12x^2$

$= 12x(18 - x) = 0$ when $x = 18$.

$\frac{d^2V}{dx^2} = 216 - 24x = -216 < 0$ when $x = 18$.

The volume is maximum when $x = 18$ in. and $y = 108 - 4(18) = 36$ in.

**30.** $V = \pi r^2 x$

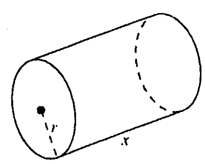
$x + 2\pi r = 108 \Rightarrow x = 108 - 2\pi r$ (see figure)

$V = \pi r^2(108 - 2\pi r) = \pi(108r^2 - 2\pi r^3)$

$\frac{dV}{dr} = \pi(216r - 6\pi r^2) = 6\pi r(36 - \pi r)$

$= 0$ when $r = \frac{36}{\pi}$ and $x = 36$.

$\frac{d^2V}{dr^2} = \pi(216 - 12\pi r) < 0$ when $r = \frac{36}{\pi}$.




Volume is maximum when $x = 36$ in. and $r = 36/\pi \approx 11.459$ in.

**31.** No. The volume will change because the shape of the container changes when squeezed.

**32.** No, there is no minimum area. If the sides are $x$ and $y$, then $2x + 2y = 20 \Rightarrow y = 10 - x$. The area is $A(x) = x(10 - x) = 10x - x^2$. This can be made arbitrarily small by selecting $x \approx 0$.

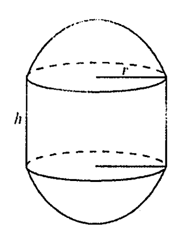
**33.** $V = 14 = \frac{4}{3}\pi r^3 + \pi r^2 h$

$h = \frac{14 - (4/3)\pi r^3}{\pi r^2} = \frac{14}{\pi r^2} - \frac{4}{3}r$

$S = 4\pi r^2 + 2\pi rh = 4\pi r^2 + 2\pi r\left(\frac{14}{\pi r^2} - \frac{4}{3}r\right) = 4\pi r^2 + \frac{28}{r} - \frac{8}{3}\pi r^2 = \frac{4}{3}\pi r^2 + \frac{28}{r}$

$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{28}{r^2} = 0$ when $r = \sqrt[3]{\frac{21}{2\pi}} \approx 1.495$ cm.

$\frac{d^2S}{dr^2} = \frac{8}{3}\pi + \frac{56}{r^3} > 0$ when $r = \sqrt[3]{\frac{21}{2\pi}}$.




The surface area is minimum when $r = \sqrt[3]{\frac{21}{2\pi}}$ cm and $h = 0$.

The resulting solid is a sphere of radius $r \approx 1.495$ cm.

**34.** $V = 4000 = \frac{4}{3}\pi r^3 + \pi r^2 h$

$$h = \frac{4000}{\pi r^2} - \frac{4}{3}r$$

Let $k$ = cost per square foot of the surface area of the sides, then $2k$ = cost per square foot of the hemispherical ends.

$$C = 2k(4\pi r^2) + k(2\pi rh) = k\left[8\pi r^2 + 2\pi r\left(\frac{4000}{\pi r^2} - \frac{4}{3}r\right)\right] = k\left[\frac{16}{3}\pi r^2 + \frac{8000}{r}\right]$$

$$\frac{dC}{dr} = k\left[\frac{32}{3}\pi r - \frac{8000}{r^2}\right] = 0 \text{ when } r = \sqrt[3]{\frac{750}{\pi}} \approx 6.204 \text{ ft and } h \approx 24.814 \text{ ft.}$$

By the Second Derivative Test, you have $\frac{d^2C}{dr^2} = k\left[\frac{32}{3}\pi + \frac{12{,}000}{r^3}\right] > 0$ when $r = \sqrt[3]{\frac{750}{\pi}}$.

The cost is minimum when $r = \sqrt[3]{\frac{750}{\pi}}$ ft and $h \approx 24.814$ ft.

**35.** Let $x$ be the length of a side of the square and $y$ the length of a side of the triangle.

$$4x + 3y = 10$$

$$A = x^2 + \frac{1}{2}y\left(\frac{\sqrt{3}}{2}y\right)$$

$$= \frac{(10 - 3y)^2}{16} + \frac{\sqrt{3}}{4}y^2$$

$$\frac{dA}{dy} = \frac{1}{8}(10 - 3y)(-3) + \frac{\sqrt{3}}{2}y = 0$$

$$-30 + 9y + 4\sqrt{3}y = 0$$

$$y = \frac{30}{9 + 4\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9 + 4\sqrt{3}}{8} > 0$$

$A$ is minimum when $y = \frac{30}{9 + 4\sqrt{3}}$ and $x = \frac{10\sqrt{3}}{9 + 4\sqrt{3}}$.

**36.** (a) Let $x$ be the side of the triangle and $y$ the side of the square.

$$A = \frac{3}{4}\left(\cot\frac{\pi}{3}\right)x^2 + \frac{4}{4}\left(\cot\frac{\pi}{4}\right)y^2 \text{ where } 3x + 4y = 20$$

$$= \frac{\sqrt{3}}{4}x^2 + \left(5 - \frac{3}{4}x\right)^2, 0 \le x \le \frac{20}{3}.$$

$$A' = \frac{\sqrt{3}}{2}x + 2\left(5 - \frac{3}{4}x\right)\left(-\frac{3}{4}\right) = 0$$

$$x = \frac{60}{4\sqrt{3} + 9}$$

When $x = 0$, $A = 25$, when $x = 60/\left(4\sqrt{3} + 9\right)$, $A \approx 10.847$, and when $x = 20/3$, $A \approx 19.245$. Area is maximum when all 20 feet are used on the square.

(b) Let $x$ be the side of the square and $y$ the side of the pentagon.

$$A = \frac{4}{4}\left(\cot\frac{\pi}{4}\right)x^2 + \frac{5}{4}\left(\cot\frac{\pi}{5}\right)y^2 \text{ where } 4x + 5y = 20$$

$$= x^2 + 1.7204774\left(4 - \frac{4}{5}x\right)^2, 0 \le x \le 5.$$

$$A' = 2x - 2.75276384\left(4 - \frac{4}{5}x\right) = 0$$

$x \approx 2.62$

When $x = 0$, $A \approx 27.528$, when $x \approx 2.62$, $A \approx 13.102$, and when $x = 5$, $A \approx 25$. Area is maximum when all 20 feet are used on the pentagon.

(c) Let $x$ be the side of the pentagon and $y$ the side of the hexagon.

$$A = \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{6}{4}\left(\cot\frac{\pi}{6}\right)y^2 \text{ where } 5x + 6y = 20$$

$$= \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{3}{2}\left(\sqrt{3}\right)\left(\frac{20 - 5x}{6}\right)^2, 0 \le x \le 4.$$

$$A' = \frac{5}{2}\left(\cot\frac{\pi}{5}\right)x + 3\sqrt{3}\left(-\frac{5}{6}\right)\left(\frac{20 - 5x}{6}\right) = 0$$

$x \approx 2.0475$

When $x = 0$, $A \approx 28.868$, when $x \approx 2.0475$, $A \approx 14.091$, and when $x = 4$, $A \approx 27.528$. Area is maximum when all 20 feet are used on the hexagon

(d) Let $x$ be the side of the hexagon and $r$ the radius of the circle.

$$A = \frac{6}{4}\left(\cot\frac{\pi}{6}\right)x^2 + \pi r^2 \text{ where } 6x + 2\pi r = 20$$

$$= \frac{3\sqrt{3}}{2}x^2 + \pi\left(\frac{10}{\pi} - \frac{3x}{\pi}\right)^2, 0 \le x \le \frac{10}{3}.$$

$$A' = 3\sqrt{3} - 6\left(\frac{10}{\pi} - \frac{3x}{\pi}\right) = 0$$

$x \approx 1.748$

When $x = 0$, $A \approx 31.831$, when $x \approx 1.748$, $A \approx 15.138$, and when $x = 10/3$, $A \approx 28.868$. Area is maximum when all 20 feet are used on the circle.

In general, using all of the wire for the figure with more sides will enclose the most area.

**37.** Let $S$ be the strength and $k$ the constant of proportionality. Given $h^2 + w^2 = 20^2$, $h^2 = 20^2 - w^2$,

$$S = kwh^2$$

$$S = kw\left(400 - w^2\right) = k\left(400w - w^3\right)$$

$$\frac{dS}{dw} = k\left(400 - 3w^2\right) = 0 \text{ when } w = \frac{20\sqrt{3}}{3} \text{ in.}$$

and $h = \dfrac{20\sqrt{6}}{3}$ in.

$$\frac{d^2S}{dw^2} = -6kw < 0 \text{ when } w = \frac{20\sqrt{3}}{3}.$$

These values yield a maximum.

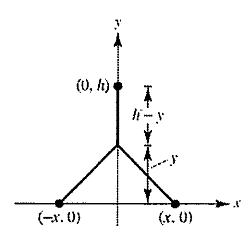
**38.** Let $A$ be the amount of the power line.

$$A = h - y + 2\sqrt{x^2 + y^2}$$

$$\frac{dA}{dy} = -1 + \frac{2y}{\sqrt{x^2 + y^2}} = 0 \text{ when } y = \frac{x}{\sqrt{3}}.$$

$$\frac{d^2A}{dy^2} = \frac{2x^2}{\left(x^2 + y^2\right)^{3/2}} > 0 \text{ for } y = \frac{x}{\sqrt{3}}.$$

The amount of power line is minimum when $y = x/\sqrt{3}$.



**39.** $C(x) = 2k\sqrt{x^2 + 4} + k(4 - x)$

$$C'(x) = \frac{2xk}{\sqrt{x^2 + 4}} - k = 0$$

$$2x = \sqrt{x^2 + 4}$$

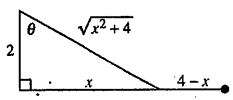
$$4x^2 = x^2 + 4$$

$$3x^2 = 4$$

$$x = \frac{2}{\sqrt{3}}$$

Or, use Exercise 50(d): $\sin\theta = \frac{C_2}{C_1} = \frac{1}{2} \Rightarrow \theta = 30°$.

So, $x = \frac{2}{\sqrt{3}}$.




**40.** $\sin\alpha = \frac{h}{s} \Rightarrow s = \frac{h}{\sin\alpha}, 0 < \alpha < \frac{\pi}{2}$

$$\tan\alpha = \frac{h}{2} \Rightarrow h = 2\tan\alpha \Rightarrow s = \frac{2\tan\alpha}{\sin\alpha} = 2\sec\alpha$$

$$I = \frac{k\sin\alpha}{s^2} = \frac{k\sin\alpha}{4\sec^2\alpha} = \frac{k}{4}\sin\alpha\cos^2\alpha$$

$$\frac{dI}{d\alpha} = \frac{k}{4}\left[\sin\alpha(-2\sin\alpha\cos\alpha) + \cos^2\alpha(\cos\alpha)\right]$$

$$= \frac{k}{4}\cos\alpha\left[\cos^2\alpha - 2\sin^2\alpha\right]$$

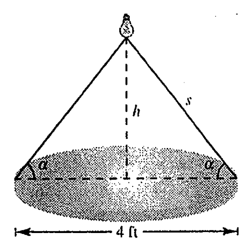
$$= \frac{k}{4}\cos\alpha\left[1 - 3\sin^2\alpha\right]$$

$$= 0 \text{ when } \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ or when } \sin\alpha = \pm\frac{1}{\sqrt{3}}.$$

Because $\alpha$ is acute, you have

$$\sin\alpha = \frac{1}{\sqrt{3}} \Rightarrow h = 2\tan\alpha = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2} \text{ ft.}$$

Because $(d^2I)/(d\alpha^2) = (k/4)\sin\alpha(9\sin^2\alpha - 7) < 0$ when $\sin\alpha = 1/\sqrt{3}$, this yields a maximum.




**41.**

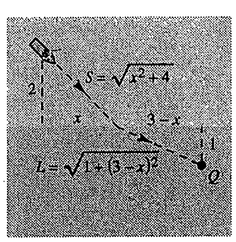
$$S = \sqrt{x^2 + 4}, L = \sqrt{1 + (3 - x)^2}$$

$$\text{Time} = T = \frac{\sqrt{x^2 + 4}}{2} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 4}} + \frac{x - 3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$\frac{x^2}{x^2 + 4} = \frac{9 - 6x + x^2}{4(x^2 - 6x + 10)}$$

$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$$




You need to find the roots of this equation in the interval $[0, 3]$. By using a computer or graphing utility you can determine that this equation has only one root in this interval $(x = 1)$. Testing at this value and at the endpoints, you see that $x = 1$ yields the minimum time. So, the man should row to a point 1 mile from the nearest point on the coast.

**42.** $T = \dfrac{\sqrt{x^2+4}}{v_1} + \dfrac{\sqrt{x^2-6x+10}}{v_2}$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2+4}} + \frac{x-3}{v_2\sqrt{x^2-6x+10}} = 0$$

Because

$$\frac{x}{\sqrt{x^2+4}} = \sin\theta_1 \text{ and } \frac{x-3}{\sqrt{x^2-6x+10}} = -\sin\theta_2$$

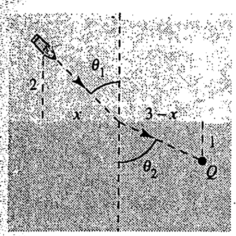
you have

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \Rightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

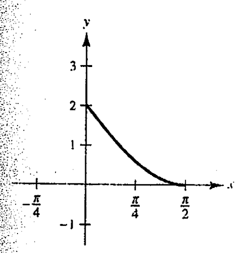
Because

$$\frac{d^2T}{dx^2} = \frac{4}{v_1(x^2+4)^{3/2}} + \frac{1}{v_2(x^2-6x+10)^{3/2}} > 0$$

this condition yields a minimum time.

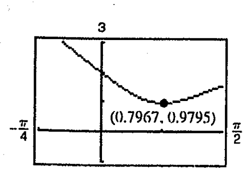


**43.** $f(x) = 2 - 2\sin x$



(a) Distance from origin to $y$-intercept is 2.
Distance from origin to $x$-intercept is $\pi/2 \approx 1.57$.

(b) $d = \sqrt{x^2+y^2} = \sqrt{x^2 + (2 - 2\sin x)^2}$




Minimum distance $= 0.9795$ at $x = 0.7967$.

(c) Let $f(x) = d^2(x) = x^2 + (2 - 2\sin x)^2$.

$f'(x) = 2x + 2(2 - 2\sin x)(-2\cos x)$

Setting $f'(x) = 0$, you obtain $x \approx 0.7967$, which corresponds to $d = 0.9795$.

**44.** $T = \dfrac{\sqrt{x^2+d_1^2}}{v_1} + \dfrac{\sqrt{d_2^2+(a-x)^2}}{v_2}$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2+d_1^2}} + \frac{x-a}{v_2\sqrt{d_2^2+(a-x)^2}} = 0$$

Because

$$\frac{x}{\sqrt{x^2+d_1^2}} = \sin\theta_1 \text{ and } \frac{x-a}{\sqrt{d_2^2+(a-x)^2}} = -\sin\theta_2$$

you have

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \Rightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

Because

$$\frac{d^2T}{dx^2} = \frac{d_1^2}{v_1(x^2+d_1^2)^{3/2}} + \frac{d_2^2}{v_2\left[d_2^2+(a-x)^2\right]^{3/2}} > 0$$

this condition yields a minimum time.

**45.** $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{144-r^2}$

$$\frac{dV}{dr} = \frac{1}{3}\pi\left[r^2\left(\frac{1}{2}\right)(144-r^2)^{-1/2}(-2r) + 2r\sqrt{144-r^2}\right]$$
$$= \frac{1}{3}\pi\left[\frac{288r - 3r^3}{\sqrt{144-r^2}}\right]$$
$$= \pi\left[\frac{r(96-r^2)}{\sqrt{144-r^2}}\right] = 0 \text{ when } r = 0, 4\sqrt{6}.$$

By the First Derivative Test, $V$ is maximum when $r = 4\sqrt{6}$ and $h = 4\sqrt{3}$.

Area of circle: $A = \pi(12)^2 = 144\pi$

Lateral surface area of cone:

$$S = \pi(4\sqrt{6})\sqrt{(4\sqrt{6})^2 + (4\sqrt{3})^2} = 48\sqrt{6}\pi$$

Area of sector:

$$144\pi - 48\sqrt{6}\pi = \frac{1}{2}\theta r^2 = 72\theta$$

$$\theta = \frac{144\pi - 48\sqrt{6}\pi}{72}$$
$$= \frac{2\pi}{3}(3 - \sqrt{6}) \approx 1.153 \text{ radians or } 66°$$

**46.** (a)

| Base 1 | Base 2 | Altitude | Area |
|---|---|---|---|
| 8 | $8 + 16 \cos 10°$ | $8 \sin 10°$ | $\approx 22.1$ |
| 8 | $8 + 16 \cos 20°$ | $8 \sin 20°$ | $\approx 42.5$ |
| 8 | $8 + 16 \cos 30°$ | $8 \sin 30°$ | $\approx 59.7$ |
| 8 | $8 + 16 \cos 40°$ | $8 \sin 40°$ | $\approx 72.7$ |
| 8 | $8 + 16 \cos 50°$ | $8 \sin 50°$ | $\approx 80.5$ |
| 8 | $8 + 16 \cos 60°$ | $8 \sin 60°$ | $\approx 83.1$ |

(b)

| Base 1 | Base 2 | Altitude | Area |
|---|---|---|---|
| 8 | $8 + 16 \cos 10°$ | $8 \sin 10°$ | $\approx 22.1$ |
| 8 | $8 + 16 \cos 20°$ | $8 \sin 20°$ | $\approx 42.5$ |
| 8 | $8 + 16 \cos 30°$ | $8 \sin 30°$ | $\approx 59.7$ |
| 8 | $8 + 16 \cos 40°$ | $8 \sin 40°$ | $\approx 72.7$ |
| 8 | $8 + 16 \cos 50°$ | $8 \sin 50°$ | $\approx 80.5$ |
| 8 | $8 + 16 \cos 60°$ | $8 \sin 60°$ | $\approx 83.1$ |
| 8 | $8 + 16 \cos 70°$ | $8 \sin 70°$ | $\approx 80.7$ |
| 8 | $8 + 16 \cos 80°$ | $8 \sin 80°$ | $\approx 74.0$ |
| 8 | $8 + 16 \cos 90°$ | $8 \sin 90°$ | $\approx 64.0$ |

The maximum cross-sectional area is approximately 83.1 $\text{ft}^2$.

(c) $A = (a + b)\dfrac{h}{2}$

$= \left[8 + (8 + 16 \cos \theta)\right]\dfrac{8 \sin \theta}{2}$

$= 64(1 + \cos \theta) \sin \theta, 0° < \theta < 90°$

(d) $\dfrac{dA}{d\theta} = 64(1 + \cos \theta) \cos \theta + (-64 \sin \theta) \sin \theta$

$= 64(\cos \theta + \cos^2 \theta - \sin^2 \theta)$

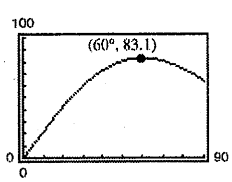
$= 64(2 \cos^2 \theta + \cos \theta - 1)$

$= 64(2 \cos \theta - 1)(\cos \theta + 1)$

$= 0$ when $\theta = 60°, 180°, 300°$.

The maximum occurs when $\theta = 60°$.

(e)




**47.** Let $d$ be the amount deposited in the bank, $i$ be the interest rate paid by the bank, and $P$ be the profit.

$P = (0.12)d - id$

$d = ki^2$ (because $d$ is proportional to $i^2$)

$P = (0.12)(ki^2) - i(ki^2) = k(0.12i^2 - i^3)$

$\dfrac{dP}{di} = k(0.24i - 3i^2) = 0$ when $i = \dfrac{0.24}{3} = 0.08$.

$\dfrac{d^2P}{di^2} = k(0.24 - 6i) < 0$ when $i = 0.08$ (**Note:** $k > 0$)

The profit is a maximum when $i = 8\%$.

**48.** (a) The profit is increasing on $(0, 40)$.

(b) The profit is decreasing on $(40, 60)$.

(c) In order to yield a maximum profit, the company should spend about \$40 thousand.

(d) The point of diminishing returns is the point where the concavity changes, which in this case is $x = 20$ thousand dollars.

**49.** $S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2$

$\dfrac{dS_1}{dm} = 2(4m - 1)(4) + 2(5m - 6)(5) + 2(10m - 3)(10)$

$= 282m - 128 = 0$ when $m = \dfrac{64}{141}$.

Line: $y = \dfrac{64}{141}x$

$S = \left|4\left(\dfrac{64}{141}\right) - 1\right| + \left|5\left(\dfrac{64}{141}\right) - 6\right| + \left|10\left(\dfrac{64}{141}\right) - 3\right|$

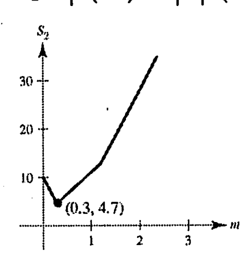
$= \left|\dfrac{256}{141} - 1\right| + \left|\dfrac{320}{141} - 6\right| + \left|\dfrac{640}{141} - 3\right| = \dfrac{858}{141} \approx 6.1$ mi

**50.** $S_2 = |4m - 1| + |5m - 6| + |10m - 3|$

Using a graphing utility, you can see that the minimum occurs when $m = 0.3$.

Line $y = 0.3x$

$S_2 = |4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3| = 4.7$ mi

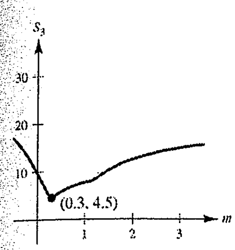


**51.** $S_3 = \dfrac{|4m - 1|}{\sqrt{m^2 + 1}} + \dfrac{|5m - 6|}{\sqrt{m^2 + 1}} + \dfrac{|10m - 3|}{\sqrt{m^2 + 1}}$

Using a graphing utility, you can see that the minimum occurs when $x \approx 0.3$.

Line: $y \approx 0.3x$

$$S_3 = \frac{|4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3|}{\sqrt{(0.3)^2 + 1}} \approx 4.5 \text{ mi.}$$




**52.** (a) Label the figure so that $r^2 = x^2 + h^2$.

Then, the area $A$ is 8 times the area of the region given by $OPQR$:

$$A = 8\left[\frac{1}{2}h^2 + (x - h)h\right] = 8\left[\frac{1}{2}(r^2 - x^2) + \left(x - \sqrt{r^2 - x^2}\right)\sqrt{r^2 - x^2}\right] = 8x\sqrt{r^2 - x^2} + 4x^2 - 4r^2$$

$$A'(x) = 8\sqrt{r^2 - x^2} - \frac{8x^2}{\sqrt{r^2 - x^2}} + 8x = 0$$

$$\frac{8x^2}{\sqrt{r^2 - x^2}} = 8x + 8\sqrt{r^2 - x^2}$$

$$x^2 = x\sqrt{r^2 - x^2} + (r^2 - x^2)$$

$$2x^2 - r^2 = x\sqrt{r^2 - x^2}$$

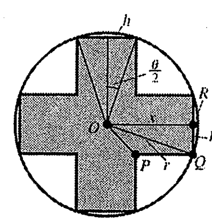
$$4x^4 - 4x^2r^2 + r^4 = x^2(r^2 - x^2)$$

$5x^4 - 5x^2r^2 + r^4 = 0$ Quadratic in $x^2$.

$$x^2 = \frac{5r^2 \pm \sqrt{25r^4 - 20r^4}}{10} = \frac{r^2}{10}\left[5 \pm \sqrt{5}\right].$$

Take positive value.

$x = r\sqrt{\dfrac{5 + \sqrt{5}}{10}} \approx 0.85065r$ Critical number




(b) Note that $\sin\dfrac{\theta}{2} = \dfrac{h}{r}$ and $\cos\dfrac{\theta}{2} = \dfrac{x}{r}$. The area $A$ of the cross equals the sum of two large rectangles minus the common square in the middle.

$$A = 2(2x)(2h) - 4h^2 = 8xh - 4h^2 = 8r^2\sin\frac{\theta}{2}\cos\frac{\theta}{2} - 4r^2\sin^2\frac{\theta}{2} = 4r^2\left(\sin\theta - \sin^2\frac{\theta}{2}\right)$$

$$A'(\theta) = 4r^2\left(\cos\theta - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = 0$$

$$\cos\theta = \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\sin\theta$$

$$\tan\theta = 2$$

$\theta = \arctan(2) \approx 1.10715$ or $63.4°$

(c) Note that $x^2 = \frac{r^2}{10}\left(5 + \sqrt{5}\right)$ and $r^2 - x^2 = \frac{r^2}{10}\left(5 - \sqrt{5}\right)$.

$$\begin{aligned} A(x) &= 8x\sqrt{r^2 - x^2} + 4x^2 - 4r^2 \\ &= 8\left[\frac{r^2}{10}\left(5 + \sqrt{5}\right)\frac{r^2}{10}\left(5 - \sqrt{5}\right)\right]^{1/2} + 4\frac{r^2}{10}\left(5 + \sqrt{5}\right) - 4r^2 \\ &= 8\left[\frac{r^4}{10}(20)\right]^{1/2} + 2r^2 + \frac{2}{5}\sqrt{5}r^2 - 4r^2 \\ &= \frac{8}{5}r^2\sqrt{5} - 2r^2 + \frac{2\sqrt{5}}{5}r^2 \\ &= 2r^2\left[\frac{4}{5}\sqrt{5} - 1 + \frac{\sqrt{5}}{5}\right] = 2r^2\left(\sqrt{5} - 1\right) \end{aligned}$$

Using the angle approach, note that $\tan\theta = 2$, $\sin\theta = \frac{2}{\sqrt{5}}$ and $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1 - \cos\theta) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)$.

So, $A(\theta) = 4r^2\left(\sin\theta - \sin^2\frac{\theta}{2}\right) = 4r^2\left(\frac{2}{\sqrt{5}} - \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\right) = \frac{4r^2\left(\sqrt{5} - 1\right)}{2} = 2r^2\left(\sqrt{5} - 1\right)$

**53.** $f(x) = x^3 - 3x;\ x^4 + 36 \le 13x^2$

$$\begin{aligned} x^4 - 13x^2 + 36 &= \left(x^2 - 9\right)\left(x^2 - 4\right) \\ &= (x - 3)(x - 2)(x + 2)(x + 3) \le 0 \end{aligned}$$

So, $-3 \le x \le -2$ or $2 \le x \le 3$.

$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$

$f$ is increasing on $(-\infty, -1)$ and $(1, \infty)$.

So, $f$ is increasing on $[-3, -2]$ and $[2, 3]$.

$f(-2) = -2$, $f(3) = 18$. The maximum value of $f$ is 18.

**54.** Let $a = \left(x + \frac{1}{x}\right)^3$ and $b = x^3 + \frac{1}{x^3}$, $x > 0$.

$$\begin{aligned} a^2 - b^2 &= \left(x + \frac{1}{x}\right)^6 - \left(x^3 + \frac{1}{x^3}\right)^2 \\ &= \left(x + \frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6} + 2\right) \end{aligned}$$

$$\begin{aligned} \text{Let } f(x) &= \frac{(x + 1/x)^6 - \left(x^6 + 1/x^6 + 2\right)}{(x + 1/x)^3 + \left(x^3 + 1/x^3\right)} \\ &= \frac{a^2 - b^2}{a + b} = a - b \\ &= \left(x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}\right) - \left(x^3 + \frac{1}{x^3}\right) \\ &= 3x + \frac{3}{x} = 3\left(x + \frac{1}{x}\right). \end{aligned}$$

Let $g(x) = x + \frac{1}{x}$, $g'(x) = 1 - \frac{1}{x^2} = 0 \Rightarrow x = 1$.

$g''(x) = \frac{2}{x^3}$ and $g''(1) = 2 > 0$. So $g$ is a minimum at $x = 1$: $g(1) = 2$.

Finally, $f$ is a minimum of $3(2) = 6$.