Section 3.7 Optimization Problems

1. (a)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$

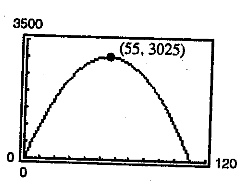
(b)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$
70	$110 - 70$	$70(110 - 70) = 2800$
80	$110 - 80$	$80(110 - 80) = 2400$
90	$110 - 90$	$90(110 - 90) = 1800$
100	$110 - 100$	$100(110 - 100) = 1000$

The maximum is attained near $x = 50$ and 60.

(c) $P = x(110 - x) = 110x - x^2$

(d)



The solution appears to be $x = 55$.

(e) $\dfrac{dP}{dx} = 110 - 2x = 0$ when $x = 55$.

$$\frac{d^2P}{dx^2} = -2 < 0$$

P is a maximum when $x = 110 - x = 55$. The two numbers are 55 and 55.

2. (a)

Height, x	Length & Width	Volume
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$
3	$24 - 2(3)$	$3[24 - 2(3)]^2 = 972$
4	$24 - 2(4)$	$4[24 - 2(4)]^2 = 1024$
5	$24 - 2(5)$	$5[24 - 2(5)]^2 = 980$
6	$24 - 2(6)$	$6[24 - 2(6)]^2 = 864$

The maximum is attained near $x = 4$.

(b) $V = x(24 - 2x)^2, 0 < x < 12$

(c) $\frac{dV}{dx} = 2x(24 - 2x)(-2) + (24 - 2x)^2 = (24 - 2x)(24 - 6x)$

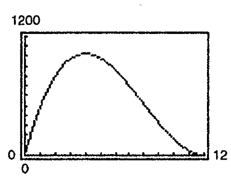
$= 12(12 - x)(4 - x) = 0$ when $x = 12, 4$ (12 is not in the domain).

$\frac{d^2V}{dx^2} = 12(2x - 16)$

$\frac{d^2V}{dx^2} < 0$ when $x = 4$.

When $x = 4, V = 1024$ is maximum.

(d)



The maximum volume seems to be 1024.

3. Let x and y be two positive numbers such that $x + y = S$.

$P = xy = x(S - x) = Sx - x^2$

$\frac{dP}{dx} = S - 2x = 0$ when $x = \frac{S}{2}$.

$\frac{d^2P}{dx^2} = -2 < 0$ when $x = \frac{S}{2}$.

P is a maximum when $x = y = S/2$.

4. Let x and y be two positive numbers such that $xy = 185$.

$S = x + y = x + \frac{185}{x}$

$\frac{dS}{dx} = 1 - \frac{185}{x^2} = 0$ when $x = \sqrt{185}$.

$\frac{d^2S}{dx^2} = \frac{370}{x^3} > 0$ when $x = \sqrt{185}$

S is a minimum when $x = y = \sqrt{185}$.

5. Let x and y be two positive numbers such that $xy = 147$.

$S = x + 3y = \frac{147}{y} + 3y$

$\frac{dS}{dy} = 3 - \frac{147}{y^2} = 0$ when $y = 7$.

$\frac{d^2S}{dy^2} = \frac{294}{y^3} > 0$ when $y = 7$.

S is minimum when $y = 7$ and $x = 21$.

6. Let x be a positive number.

$S = x + \frac{1}{x}$

$\frac{dS}{dx} = 1 - \frac{1}{x^2} = 0$ when $x = 1$.

$\frac{d^2S}{dx^2} = \frac{2}{x^3} > 0$ when $x = 1$.

The sum is a minimum when $x = 1$ and $1/x = 1$.

7. Let x and y be two positive numbers such that $x + 2y = 108$.

$$P = xy = y(108 - 2y) = 108y - 2y^2$$

$$\frac{dP}{dy} = 108 - 4y = 0 \text{ when } y = 27.$$

$$\frac{d^2P}{dy^2} = -4 < 0 \text{ when } y = 27.$$

P is a maximum when $x = 54$ and $y = 27$.

8. Let x and y be two positive numbers such that $x^2 + y = 54$.

$$P = xy = x(54 - x^2) = 54x - x^3$$

$$\frac{dP}{dx} = 54 - 3x^2 = 0 \text{ when } x = 3\sqrt{2}.$$

$$\frac{d^2P}{dx^2} = -6x < 0 \text{ when } x = 3\sqrt{2}.$$

The product is a maximum when $x = 3\sqrt{2}$ and $y = 36$.

9. Let x be the length and y the width of the rectangle.

$$2x + 2y = 80$$

$$y = 40 - x$$

$$A = xy = x(40 - x) = 40x - x^2$$

$$\frac{dA}{dx} = 40 - 2x = 0 \text{ when } x = 20.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = 20.$$

A is maximum when $x = y = 20$ m.

10. Let x be the length and y the width of the rectangle.

$$2x + 2y = P$$

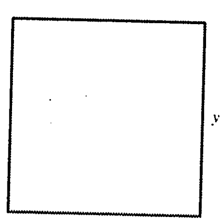
$$y = \frac{P - 2x}{2} = \frac{P}{2} - x$$

$$A = xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$$

$$\frac{dA}{dx} = \frac{P}{2} - 2x = 0 \text{ when } x = \frac{P}{4}.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = \frac{P}{4}.$$

A is maximum when $x = y = P/4$ units. (A square!)



11. Let x be the length and y the width of the rectangle.

$$xy = 32$$

$$y = \frac{32}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{32}{x}\right) = 2x + \frac{64}{x}$$

$$\frac{dP}{dx} = 2 - \frac{64}{x^2} = 0 \text{ when } x = 4\sqrt{2}.$$

$$\frac{d^2P}{dx^2} = \frac{128}{x^3} > 0 \text{ when } x = 4\sqrt{2}.$$

P is minimum when $x = y = 4\sqrt{2}$ ft.

12. Let x be the length and y the width of the rectangle.

$$xy = A$$

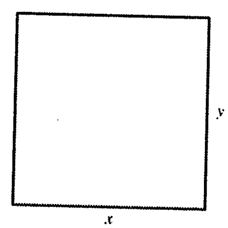
$$y = \frac{A}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{A}{x}\right) = 2x + \frac{2A}{x}$$

$$\frac{dP}{dx} = 2 - \frac{2A}{x^2} = 0 \text{ when } x = \sqrt{A}.$$

$$\frac{d^2P}{dx^2} = \frac{4A}{x^3} > 0 \text{ when } x = \sqrt{A}.$$

P is minimum when $x = y = \sqrt{A}$ cm. (A square!)



13. $d = \sqrt{(x - 2)^2 + \left[x^2 - (1/2)\right]^2}$

$= \sqrt{x^4 - 4x + (17/4)}$

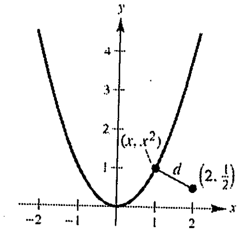
Because d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^4 - 4x + \tfrac{17}{4}.$$

$$f'(x) = 4x^3 - 4 = 0$$

$$x = 1$$

By the First Derivative Test, the point nearest to $\left(2, \frac{1}{2}\right)$ is $(1, 1)$.



14. $f(x) = (x-1)^2, (-5, 3)$

$$d = \sqrt{(x+5)^2 + \left[(x-1)^2 - 3\right]^2}$$
$$= \sqrt{(x^2 + 10x + 25) + (x^2 - 2x - 2)^2}$$
$$= \sqrt{(x^2 + 10x + 25) + (x^4 - 4x^3 + 8x + 4)}$$
$$= \sqrt{x^4 - 4x^3 + x^2 + 18x + 29}$$

Because d is smallest when the expression inside the radical is smallest, you need to find the critical numbers of

$$g(x) = x^4 - 4x^3 + x^2 + 18x + 29$$
$$g'(x) = 4x^3 - 12x^2 + 2x + 18$$
$$= 2(x+1)(2x^2 - 8x + 9) = 0$$
$$x = -1$$

By the First Derivative Test, $x = -1$ yields a minimum. So, $(-1, 4)$ is closest to $(-5, 3)$.

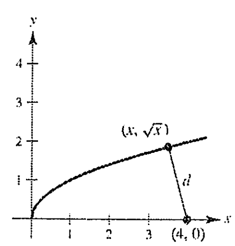
15. $d = \sqrt{(x-4)^2 + (\sqrt{x} - 0)^2}$

$$= \sqrt{x^2 - 7x + 16}$$

Because d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^2 - 7x + 16.$$
$$f'(x) = 2x - 7 = 0$$
$$x = \tfrac{7}{2}$$

By the First Derivative Test, the point nearest to $(4, 0)$ is $\left(7/2, \sqrt{7/2}\right)$.



16. $f(x) = \sqrt{x-8}, (12, 0)$

$$d = \sqrt{(x-12)^2 + \left(\sqrt{x-8} - 0\right)^2}$$
$$= \sqrt{x^2 - 24x + 144 + x - 8}$$
$$= \sqrt{x^2 - 23x + 136}$$

Because d is smallest when the expression inside the radical is smallest, you need to find the critical numbers of

$$g(x) = x^2 - 23x + 136$$
$$g'(x) = 2x - 23 = 0 \text{ when } x = \frac{23}{2}$$
$$g''(x) = 2 > 0 \text{ at } x = \frac{23}{2}$$

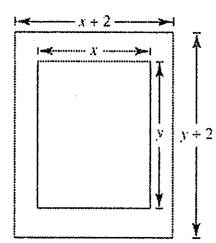
The point nearest to $(12, 0)$ is

$$\left(\frac{23}{2}, f\left(\frac{23}{2}\right)\right) = \left(\frac{23}{2}, \frac{\sqrt{14}}{2}\right)$$

17. $xy = 30 \Rightarrow y = \dfrac{30}{x}$

$$A = (x+2)\left(\frac{30}{x} + 2\right) \text{ (see figure)}$$
$$\frac{dA}{dx} = (x+2)\left(\frac{-30}{x^2}\right) + \left(\frac{30}{x} + 2\right)$$
$$= \frac{2(x^2 - 30)}{x^2} = 0 \text{ when } x = \sqrt{30}.$$
$$y = \frac{30}{\sqrt{30}} = \sqrt{30}$$

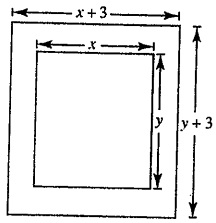
By the First Derivative Test, the dimensions $(x+2)$ by $(y+2)$ are $\left(2 + \sqrt{30}\right)$ by $\left(2 + \sqrt{30}\right)$ (approximately 7.477 by 7.477). These dimensions yield a minimum area.



18. $xy = 36 \Rightarrow y = \dfrac{36}{x}$

$$A = (x+3)(y+3) = (x+3)\left(\frac{36}{x} + 3\right)$$
$$= 36 + \frac{108}{x} + 3x + 9$$
$$\frac{dA}{dx} = \frac{-108}{x^2} + 3 = 0 \Rightarrow 3x^2 = 108 \Rightarrow x = 6, y = 6$$

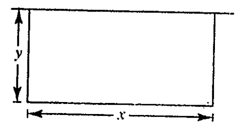
Dimensions: 9×9



19. $xy = 245{,}000$ (see figure)

$$S = x + 2y$$
$$= \left(x + \frac{490{,}000}{x}\right) \text{ where } S \text{ is the length of fence needed.}$$
$$\frac{dS}{dx} = 1 - \frac{490{,}000}{x^2} = 0 \text{ when } x = 700.$$
$$\frac{d^2S}{dx^2} = \frac{980{,}000}{x^3} > 0 \text{ when } x = 700.$$

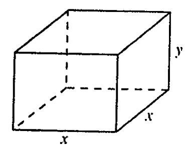
S is a minimum when $x = 700$ m and $y = 350$ m.



20. $S = 2x^2 + 4xy = 337.5$

$$y = \frac{337.5 - 2x^2}{4x}$$
$$V = x^2y = x^2\left[\frac{337.5 - 2x^2}{4x}\right] = 84.375x - \frac{1}{2}x^3$$
$$\frac{dV}{dx} = 84.375 - \frac{3}{2}x^2 = 0 \Rightarrow x^2 = 56.25 \Rightarrow x = 7.5 \text{ and } y = 7.5.$$
$$\frac{d^2V}{dx^2} = -3x < 0 \text{ for } x = 7.5.$$

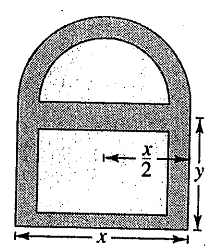
The maximum value occurs when $x = y = 7.5$ cm.



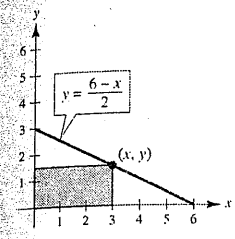
21. $16 = 2y + x + \pi\left(\dfrac{x}{2}\right)$

$$32 = 4y + 2x + \pi x$$
$$y = \frac{32 - 2x - \pi x}{4}$$
$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \left(\frac{32 - 2x - \pi x}{4}\right)x + \frac{\pi x^2}{8} = 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$
$$\frac{dA}{dx} = 8 - x - \frac{\pi}{2}x + \frac{\pi}{4}x = 8 - x\left(1 + \frac{\pi}{4}\right) = 0 \text{ when } x = \frac{8}{1 + (\pi/4)} = \frac{32}{4 + \pi}.$$
$$\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0 \text{ when } x = \frac{32}{4 + \pi}.$$
$$y = \frac{32 - 2[32/(4+\pi)] - \pi[32/(4+\pi)]}{4} = \frac{16}{4 + \pi}$$

The area is maximum when $y = \dfrac{16}{4 + \pi}$ ft and $x = \dfrac{32}{4 + \pi}$ ft.



22. You can see from the figure that $A = xy$ and $y = \dfrac{6 - x}{2}$.



$$A = x\left(\frac{6 - x}{2}\right) = \frac{1}{2}\left(6x - x^2\right).$$

$$\frac{dA}{dx} = \frac{1}{2}(6 - 2x) = 0 \text{ when } x = 3.$$

$$\frac{d^2A}{dx^2} = -1 < 0 \text{ when } x = 3.$$

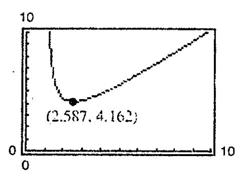
A is a maximum when $x = 3$ and $y = 3/2$.

23. (a) $\dfrac{y - 2}{0 - 1} = \dfrac{0 - 2}{x - 1}$

$$y = 2 + \frac{2}{x - 1}$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(2 + \frac{2}{x - 1}\right)^2} = \sqrt{x^2 + 4 + \frac{8}{x - 1} + \frac{4}{(x - 1)^2}}, \quad x > 1$$

(b)



L is minimum when $x \approx 2.587$ and $L \approx 4.162$.

(c) Area $= A(x) = \dfrac{1}{2}xy = \dfrac{1}{2}x\left(2 + \dfrac{2}{x - 1}\right) = x + \dfrac{x}{x - 1}$

$$A'(x) = 1 + \frac{(x - 1) - x}{(x - 1)^2} = 1 - \frac{1}{(x - 1)^2} = 0$$

$$(x - 1)^2 = 1$$

$$x - 1 = \pm 1$$

$$x = 0, 2 \ (\text{select } x = 2)$$

They $y = 4$ and $A = 4$.

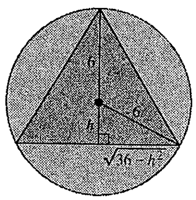
Vertices: $(0, 0), (2, 0), (0, 4)$

24. (a) $A = \frac{1}{2}\text{ base} \times \text{height} = \frac{1}{2}\left(2\sqrt{36 - h^2}\right)(6 + h) = \sqrt{36 - h^2}(6 + h)$

$$\frac{dA}{dh} = \frac{1}{2}\left(36 - h^2\right)^{-1/2}(-2h)(6 + h) + \left(36 - h^2\right)^{1/2}$$

$$= \left(36 - h^2\right)^{-1/2}\left[-h(6 + h) + \left(36 - h^2\right)\right] = \frac{-2\left(h^2 + 3h - 18\right)}{\sqrt{36 - h^2}} = \frac{-2(h + 6)(h - 3)}{\sqrt{36 - h^2}}$$

$\frac{dA}{dh} = 0$ when $h = 3$, which is a maximum by the First Derivative Test. So, the sides are $2\sqrt{36 - h^2} = 6\sqrt{3}$, an equilateral triangle. Area $= 27\sqrt{3}$ sq. units.



(b)

$$\cos\alpha = \frac{6 + h}{2\sqrt{3}\sqrt{6 + h}} = \frac{\sqrt{6 + h}}{2\sqrt{3}}$$

$$\tan\alpha = \frac{\sqrt{36 - h^2}}{6 + h}$$

$$\text{Area} = 2\left(\frac{1}{2}\right)\left(\sqrt{36 - h^2}\right)(6 + h) = (6 + h)^2\tan\alpha = 144\cos^4\alpha\tan\alpha$$

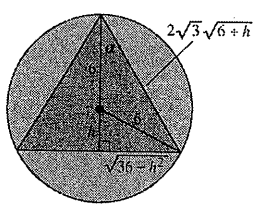
$$A'(\alpha) = 144\left[\cos^4\alpha\sec^2\alpha + 4\cos^3(-\sin\alpha)\tan\alpha\right] = 0$$

$$\Rightarrow \cos^4\alpha\sec^2\alpha = 4\cos^3\alpha\sin\alpha\tan\alpha$$

$$1 = 4\cos\alpha\sin\alpha\tan\alpha$$

$$\frac{1}{4} = \sin^2\alpha$$

$$\sin\alpha = \frac{1}{2} \Rightarrow \alpha = 30° \text{ and } A = 27\sqrt{3}.$$



(c) Equilateral triangle

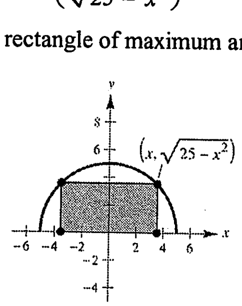
25. $A = 2xy = 2x\sqrt{25 - x^2}$ (see figure)

$$\frac{dA}{dx} = 2x\left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{25 - x^2}}\right) + 2\sqrt{25 - x^2} = 2\left(\frac{25 - 2x^2}{\sqrt{25 - x^2}}\right) = 0 \text{ when } x = y = \frac{5\sqrt{2}}{2} \approx 3.54.$$

By the First Derivative Test, the inscribed rectangle of maximum area has vertices

$$\left(\pm\frac{5\sqrt{2}}{2}, 0\right), \left(\pm\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right).$$

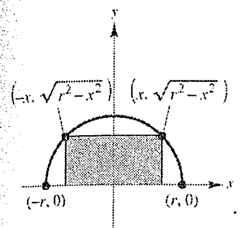
Width: $\frac{5\sqrt{2}}{2}$; Length: $5\sqrt{2}$



26. $A = 2xy = 2x\sqrt{r^2 - x^2}$ (see figure)

$$\frac{dA}{dx} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} = 0 \text{ when } x = \frac{\sqrt{2}r}{2}.$$

By the First Derivative Test, A is maximum when the rectangle has dimensions $\sqrt{2}r$ by $(\sqrt{2}r)/2$.

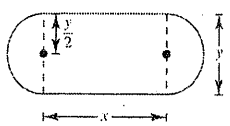


27. (a) $P = 2x + 2\pi r$

$$= 2x + 2\pi\left(\frac{y}{2}\right)$$

$$= 2x + \pi y = 200$$

$$\Rightarrow y = \frac{200 - 2x}{\pi} = \frac{2}{\pi}(100 - x)$$



(b)

Length, x	Width, y	Area, xy
10	$\frac{2}{\pi}(100 - 10)$	$(10)\frac{2}{\pi}(100 - 10) \approx 573$
20	$\frac{2}{\pi}(100 - 20)$	$(20)\frac{2}{\pi}(100 - 20) \approx 1019$
30	$\frac{2}{\pi}(100 - 30)$	$(30)\frac{2}{\pi}(100 - 30) \approx 1337$
40	$\frac{2}{\pi}(100 - 40)$	$(40)\frac{2}{\pi}(100 - 40) \approx 1528$
50	$\frac{2}{\pi}(100 - 50)$	$(50)\frac{2}{\pi}(100 - 50) \approx 1592$
60	$\frac{2}{\pi}(100 - 60)$	$(60)\frac{2}{\pi}(100 - 60) \approx 1528$

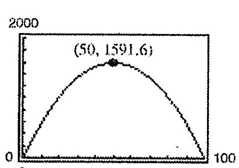
The maximum area of the rectangle is approximately 1592 m^2.

(c) $A = xy = x\frac{2}{\pi}(100 - x) = \frac{2}{\pi}(100x - x^2)$

(d) $A' = \frac{2}{\pi}(100 - 2x)$. $A' = 0$ when $x = 50$.

Maximum value is approximately 1592 when length = 50 m and width = $\frac{100}{\pi}$.

(e)



Maximum area is approximately 1591.55 m^2 ($x = 50$ m).

28. $V = \pi r^2 h = 22$ cubic inches or $h = \dfrac{22}{\pi r^2}$

(a)

Radius, r	Height	Surface Area
0.2	$\dfrac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\dfrac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\dfrac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \dfrac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\dfrac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \dfrac{22}{\pi(0.8)^2}\right] \approx 59.0$

(b)

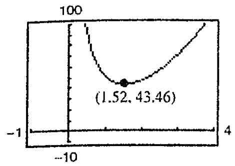
Radius, r	Height	Surface Area
0.2	$\dfrac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\dfrac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\dfrac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \dfrac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\dfrac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \dfrac{22}{\pi(0.8)^2}\right] \approx 59.0$
1.0	$\dfrac{22}{\pi(1.0)^2}$	$2\pi(1.0)\left[1.0 + \dfrac{22}{\pi(1.0)^2}\right] \approx 50.3$
1.2	$\dfrac{22}{\pi(1.2)^2}$	$2\pi(1.2)\left[1.2 + \dfrac{22}{\pi(1.2)^2}\right] \approx 45.7$
1.4	$\dfrac{22}{\pi(1.4)^2}$	$2\pi(1.4)\left[1.4 + \dfrac{22}{\pi(1.4)^2}\right] \approx 43.7$
1.6	$\dfrac{22}{\pi(1.6)^2}$	$2\pi(1.6)\left[1.6 + \dfrac{22}{\pi(1.6)^2}\right] \approx 43.6$
1.8	$\dfrac{22}{\pi(1.8)^2}$	$2\pi(1.8)\left[1.8 + \dfrac{22}{\pi(1.8)^2}\right] \approx 44.8$
2.0	$\dfrac{22}{\pi(2.0)^2}$	$2\pi(2.0)\left[2.0 + \dfrac{22}{\pi(2.0)^2}\right] \approx 47.1$

The minimum seems to the about 43.6 for $r = 1.6$.

(c) $S = 2\pi r^2 + 2\pi rh$

$$= 2\pi r(r + h) = 2\pi r\left[r + \frac{22}{\pi r^2}\right] = 2\pi r^2 + \frac{44}{r}$$

(d)



The minimum seems to be 43.46 for $r \approx 1.52$.

(e) $\dfrac{dS}{dr} = 4\pi r - \dfrac{44}{r^2} = 0$ when $r = \sqrt[3]{11/\pi} \approx 1.52$ in.

$h = \dfrac{22}{\pi r^2} \approx 3.04$ in.

Note: Notice that $h = \dfrac{22}{\pi r^2} = \dfrac{22}{\pi(11/\pi)^{2/3}} = 2\left(\dfrac{11^{1/3}}{\pi^{1/3}}\right) = 2r$.

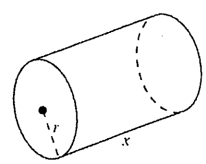
29. Let x be the sides of the square ends and y the length of the package.

$$P = 4x + y = 108 \Rightarrow y = 108 - 4x$$
$$V = x^2y = x^2(108 - 4x) = 108x^2 - 4x^3$$
$$\frac{dV}{dx} = 216x - 12x^2$$
$$= 12x(18 - x) = 0 \text{ when } x = 18.$$
$$\frac{d^2V}{dx^2} = 216 - 24x = -216 < 0 \text{ when } x = 18.$$

The volume is maximum when $x = 18$ in. and $y = 108 - 4(18) = 36$ in.

30.

$$V = \pi r^2 x$$
$$x + 2\pi r = 108 \Rightarrow x = 108 - 2\pi r \text{ (see figure)}$$
$$V = \pi r^2(108 - 2\pi r) = \pi(108r^2 - 2\pi r^3)$$
$$\frac{dV}{dr} = \pi(216r - 6\pi r^2) = 6\pi r(36 - \pi r)$$
$$= 0 \text{ when } r = \frac{36}{\pi} \text{ and } x = 36.$$
$$\frac{d^2V}{dr^2} = \pi(216 - 12\pi r) < 0 \text{ when } r = \frac{36}{\pi}.$$



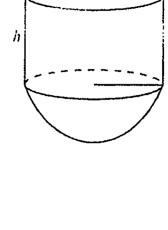
Volume is maximum when $x = 36$ in. and $r = 36/\pi \approx 11.459$ in.

31. No. The volume will change because the shape of the container changes when squeezed.

32. No, there is no minimum area. If the sides are x and y, then $2x + 2y = 20 \Rightarrow y = 10 - x$. The area is $A(x) = x(10 - x) = 10x - x^2$. This can be made arbitrarily small by selecting $x \approx 0$.

33.

$$V = 14 = \frac{4}{3}\pi r^3 + \pi r^2 h$$
$$h = \frac{14 - (4/3)\pi r^3}{\pi r^2} = \frac{14}{\pi r^2} - \frac{4}{3}r$$
$$S = 4\pi r^2 + 2\pi rh = 4\pi r^2 + 2\pi r\left(\frac{14}{\pi r^2} - \frac{4}{3}r\right) = 4\pi r^2 + \frac{28}{r} - \frac{8}{3}\pi r^2 = \frac{4}{3}\pi r^2 + \frac{28}{r}$$
$$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{28}{r^2} = 0 \text{ when } r = \sqrt[3]{\frac{21}{2\pi}} \approx 1.495 \text{ cm.}$$
$$\frac{d^2S}{dr^2} = \frac{8}{3}\pi + \frac{56}{r^3} > 0 \text{ when } r = \sqrt[3]{\frac{21}{2\pi}}.$$



The surface area is minimum when $r = \sqrt[3]{\dfrac{21}{2\pi}}$ cm and $h = 0$.

The resulting solid is a sphere of radius $r \approx 1.495$ cm.