4.1b Trig Integrals and Differential Equations

Review:

1) $\frac{d}{dx}\sin u = \cos u \cdot u'$
2) $\frac{d}{dx}\cos u = -\sin u \cdot u'$
3) $\frac{d}{dx}\tan u = \sec^2 u \cdot u'$
4) $\frac{d}{dx}\cot u = -\csc^2 u \cdot u'$
5) $\frac{d}{dx}\sec u = \sec u \tan u \cdot u'$
6) $\frac{d}{dx}\csc u = -\csc u \cot u \cdot u'$

Trig Integrals

1) $\int \sin u\,du = -\cos u + C$
2) $\int \cos u\,du = \sin u + C$
3) $\int \sec^2 u\,du = \tan u + C$
4) $\int \csc^2 u\,du = -\cot u + C$
5) $\int \sec u \tan u\,du = \sec u + C$
6) $\int \csc u \cot u\,du = -\csc u + C$

Ex. 1 $\int (1+\cot^2 x)dx$

*Use trig identity to rewrite integrand: $1+\cot^2 x = \csc^2 x$

$$\int \csc^2 x\,dx = -\cot x + C$$

Ex. 2 $\int \frac{\sin x}{\cos^2 x}dx$

* Try rewriting in different forms to match integral formula

$$\int \frac{\sin x}{\cos x \cdot \cos x}dx$$

$$\int \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}dx$$

$$\int \tan x \sec x\,dx$$

$$= \boxed{\sec x + C}$$

4.1b (continued)

Differential Equations: These are simply equations that involves derivatives.

Steps:

1) Rewrite y' as $\frac{dy}{dx}$
2) Separate variables on either side of equation
3) Take Integral of both sides.
4) Solve for C if finding specific differential equation.

$\boxed{\text{Ex.3}}$ Suppose $y' = 2$. Solve for y.

$$\frac{dy}{dx} = 2$$

$$dy = 2dx$$

$$\int dy = \int 2dx$$

$$y = 2x + C$$

we only have to write "+C" once on right side of equation.

$\boxed{\text{Ex.4}}$ Solve this General differential equation (like ex.3)

Solve for y: $\frac{dy}{dx} = x^3$

$$dy = x^3 dx$$

$$\int dy = \int x^3 dx$$

$$\boxed{y = \frac{x^4}{4} + c}$$

$\boxed{\text{Ex.5}}$ Solve this Specific differential equation

Solve for y: $y' = 3x - 4$ and the point $(4,10)$ is on graph of y.

$$\frac{dy}{dx} = 3x - 4$$

$$dy = 3x - 4\,dx$$

$$\int dy = \int 3x - 4\,dx$$

$$y = \frac{3x^2}{2} - 4x + C$$ ← plug in

$$10 = \frac{3(4)^2}{2} - 4(4) + c$$

$$10 = 3(8) - 16 + c$$

$$\underline{\underline{2 = C}}$$

$$\boxed{y = \frac{3}{2}x^2 - 4x + 2}$$

4.16 (continued) More diff. equation examples

Ex. 6 Suppose $f''(x)=6x+4$, $f'(0)=3$, and $f(1)=5$.
Find $f(x)$.

* To help distinguish the constants of integration, use "+c" for the first constant and use "+K" for the second constant of integration.

$f''(x)=6x+4$

$f'(x)=\frac{6x^2}{2}+4x+c$ use $f'(0)=3$ to solve for c.

$3=\frac{6(0)^2}{2}+4(0)+c$

$\underline{\underline{3=c}} \rightarrow f'(x)=3x^2+4x+3$

$f'(x)=3x^2+4x+3$

$f(x)=\frac{3x^3}{3}+\frac{4x^2}{2}+3x+K$ Use $f(1)=5$ to find K.

$f(x)=x^3+2x^2+3x+K$

$5=1^3+2(1)^2+3(1)+K$

$5=6+K$

$\underline{\underline{-1=K}} \longrightarrow \boxed{f(x)=x^3+2x^2+3x-1}$

Ex. 7 Given $g''(x)=12x+6$ and $g(0)=4$ and $g(1)=-2$. Find $g(x)$.

$g'(x)=\frac{12x^2}{2}+6x+c$

$g(x)=\frac{6x^3}{3}+\frac{6x^2}{2}+cx+K$

$g(x)=2x^3+3x^2+cx+K$

$4=2(0)^3+3(0)^2+c(0)+K$

$\underline{4=K}$

$g(x)=2x^3+3x^2+cx+4$

$-2=2(1)^3+3(1)^2+c(1)+4$

$-2=2+3+c+4$

$-2=9+c$

$\underline{-11=c}$

$\boxed{g(x)=2x^3+3x^2-11x+4}$

4.1b p. 255-256 #5, 7, 35-41 odd, 47, 48, 55-61 odd

Find the general solution of the differential equation.

5) $\frac{dy}{dt} = 3t^2$ | $\int dy = \int 3t^2 dt$ | $\boxed{y = t^3 + c}$ | Check solution:

$dy = 3t^2 dt$ | $y = 3\left(\frac{t^3}{3}\right) + c$ | $\frac{d}{dt}(t^3) = 3t^2$ ✓

7) $\frac{dy}{dx} = x^{3/2}$ | $\int dy = \int x^{3/2} dx$ | $\boxed{y = \frac{2}{5}x^{5/2} + c}$ | check: $\frac{d}{dx}\left(\frac{2}{5}x^{5/2} + c\right) = \frac{2}{5} \cdot \frac{5}{2} x^{5/2 - 2/2}$

$dy = x^{3/2} dx$ | $y = \frac{x^{5/2}}{5/2} + c$ | $= x^{3/2}$ ✓

35) $\int (2\sin x + 3\cos x) dx$ | $= 2(-\cos x) + 3(\sin x) + c$ | check: $\frac{d}{dx}(-2\cos x + 3\sin x)$

$= 2\int \sin x\, dx + 3\int \cos x\, dx$ | $= \boxed{-2\cos x + 3\sin x + c}$ | $= -2(-\sin x) + 3\cos x$

$= 2\sin x + 3\cos x$ ✓

37) $\int (1 - \csc t \cot t) dt$ | $t - (-\csc t) + c$ | check: $\frac{d}{dt}(t + \csc t)$

$= \int 1 dt - \int \csc t \cot t\, dt$ | $= \boxed{t + \csc t + c}$ | $= 1 - \csc t \cot t$ ✓

41) $\int \tan^2 y + 1\, dy$ | $= \boxed{\tan y + c}$ | check: $\frac{d}{dy}(\tan y) = \sec^2 y = 1 + \tan^2 y$ ✓

use trig identity to rewrite problem

$\int \sec^2 y\, dy$

48) Find equation for y, given the derivative: $\frac{dy}{dx} = 2(x-1)$ at $(3,2)$

$dy = 2(x-1)dx$

$\int dy = \int 2(x-1)dx$

$\int dy = \int 2x - 2dx$

$y = \frac{2x^2}{2} - 2x + c$

$y = x^2 - 2x + c$

*plug in (3,2) to solve for c.

$2 = (3)^2 - 2(3) + c$

$2 = 9 - 6 + c$

$2 = 3 + c$

$-1 = c$

$\boxed{y = x^2 - 2x - 1}$

55) Solve the differential equation. $f'(x) = 4x$, $f(0) = 6$

$\frac{dy}{dx} = 4x$

$dy = 4x\,dx$

$\int dy = \int 4x\,dx$

$y = \frac{4x^2}{2} + c$

$y = 2x^2 + c$

*plug in (0,6)

$6 = 2(0)^2 + c$

$6 = c$

$\boxed{y = 2x^2 + 6}$

59) $f''(x) = 2$, $f'(2) = 5$, $f(2) = 10$

$f'(x) = \int 2\,dx$

$f'(x) = 2x + C$

*Use $f'(2) = 5$ to solve for c

$5 = 2(2) + C$

$5 = 4 + C$

$1 = C$

$f'(x) = 2x + 1$

$f(x) = \int 2x + 1dx$

$f(x) = \frac{2x^2}{2} + x + k$

$f(x) = x^2 + x + k$

*plug in $f(2) = 10$ to solve for k.

$10 = 2^2 + 2 + k$

$10 = 4 + 2 + k$

$10 = 6 + k$

$4 = k$

$\boxed{f(x) = x^2 + x + 4}$

61) $f''(x) = x^{-3/2}$ $f'(4) = 2$, $f(0) = 0$

$f'(x) = \int x^{-3/2}dx$

$f'(x) = \frac{x^{-1/2}}{-1/2} + c$

$f'(x) = -2x^{-1/2} + c$

$2 = -2(4)^{-1/2} + c$

$2 = \frac{-2}{\sqrt{4}} + c$

$2 = \frac{-2}{2} + c$

$2 = -1 + c$

$3 = c$

$f'(x) = -2x^{-1/2} + 3$

$f(x) = \int -2x^{-1/2} + 3dx$

$f(x) = \frac{-2x^{1/2}}{1/2} + 3x + K$

$f(x) = -4x^{1/2} + 3x + K$

$0 = -4(0) + 3(0) + K$

$0 = K$

$\boxed{f(x) = -4x^{1/2} + 3x}$