# Ch. 4.2a Notes

Key

## I. Sigma Notation

upper bound

$$\sum_{i=2}^{5} a_i = a_2 + a_3 + a_4 + a_5$$

index ; lower bound

**Ex. 1** $\displaystyle\sum_{i=2}^{4} i^2+1 = (2^2+1)+(3^2+1)+(4^2+1) = \boxed{32}$

## II. Summation Formulas *Memorize these*

1) $\displaystyle\sum_{i=1}^{n} 1 = n$

2) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2}+\frac{n}{2}$

3) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$

4) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$

5) $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

**Ex. 2** $\displaystyle\sum_{i=1}^{8}(3i^2+2)$

$$= \sum_{i=1}^{8} 3i^2 + \sum_{i=1}^{8} 2$$

$$= 3\sum_{i=1}^{8} i^2 + 2\sum_{i=1}^{8} 1$$

$$= 3\cdot\frac{8(9)(17)}{6} + 2\cdot 8$$

$$= 612 + 16$$

$$= \boxed{628}$$

**Ex. 3** $\displaystyle\sum_{i=1}^{10}(i+2)^2$

$$= \sum_{i=1}^{10} i^2+4i+4$$

$$= \sum_{i=1}^{10} i^2 + 4\sum_{i=1}^{10} i + 4\sum_{i=1}^{10} 1$$

$$= \frac{10(11)(21)}{6} + 4\cdot\frac{10(11)}{2} + 4(10)$$

$$= 385 + 220 + 40$$

$$= \boxed{645}$$

**Ex. 4** $\displaystyle\sum_{k=1}^{n}\frac{1}{n}(k^2-1)$

$$= \frac{1}{n}\sum_{k=1}^{n} k^2 - \frac{1}{n}\sum_{k=1}^{n} 1$$

$$= \frac{1}{\cancel{n}}\cdot\frac{\cancel{n}(n+1)(2n+1)}{6} - \frac{1}{\cancel{n}}(\cancel{n})$$

$$= \frac{(n+1)(2n+1)}{6} - 1$$

$$= \frac{2n^2+3n+1}{6} - \frac{6}{6}$$

$$= \boxed{\frac{2n^2+3n-5}{6}}$$

4.2a Notes (continued)

III. Limit as $n$ approaches infinity

* Think back about finding horizontal asymptotes.

$\boxed{\text{Ex. 5}}$ If $s(n)=\frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$ then find $\lim\limits_{n\to\infty} s(n)$

$$=\lim_{n\to\infty}\frac{n^2+\cancel{n}}{2n^2}=\boxed{\frac{1}{2}}$$

← Take coefficient if degrees are same in numerator and denominator

$\boxed{\text{Ex. 6}}$ Find $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{4i}{n^2}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}4i$$

$$=\lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n}i$$

$$=\lim_{n\to\infty}\frac{4}{n^2}\cdot\frac{n(n+1)}{2}=\lim_{n\to\infty}\frac{4n^2+4n}{2n^2}$$

$$=\frac{4}{2}=\boxed{2}$$

$\boxed{\text{Ex. 7}}$ Find $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(1+\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{4i}{n}+\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2}{n}+\frac{8i}{n^2}+\frac{8i^2}{n^3}\right)$$

$$=\lim_{n\to\infty}\left[\frac{2}{n}\sum_{i=1}^{n}1+\frac{8}{n^2}\sum_{i=1}^{n}i+\frac{8}{n^3}\sum_{i=1}^{n}i^2\right]$$

$$=\lim_{n\to\infty}\left[\frac{2}{n}(n)+\frac{8}{n^{\cancel{2}}}\cdot\frac{\cancel{n}(n+1)}{2}+\frac{8}{n^{\cancel{3}}}\cdot\frac{\cancel{n}(n+1)(2n+1)}{6}\right]$$

$$=\lim_{n\to\infty}\left[\frac{2n}{n}+\frac{8}{n}\cdot\frac{n+1}{2}+\frac{8}{n^2}\cdot\frac{2n^2+3n+1}{6}\right]$$

$$=\lim_{n\to\infty}\left[\frac{2n}{n}+\frac{8n+1}{2n}+\frac{16n^2+24n+8}{6n^2}\right]$$

$$=\frac{2}{1}+\frac{8}{2}+\frac{16}{6}$$

$$=2+4+8/3=\boxed{\frac{26}{3}}$$

Reminder:

* $\boxed{n \text{ is a constant}}$ *

# 4.2a Selected HW problems

p. 267–268 #1–21 odd, 31, 33, 39, 41, 43

9) Use sigma notation to write the sum:

$$\left[5\left(\tfrac{1}{8}\right)+3\right]+\left[5\left(\tfrac{2}{8}\right)+3\right]+\dots+\left[5\left(\tfrac{8}{8}\right)+3\right]$$

* Notice the incrementing values

$$\boxed{\sum_{i=1}^{8} 5\left(\frac{i}{8}\right)+3}$$

11) $$\left[\left(\frac{2}{n}\right)^3-\frac{2}{n}\right]\left(\frac{2}{n}\right)+\dots\left[\left(\frac{2(2)}{n}\right)^3-\frac{2(2)}{n}\right]\left(\frac{2}{n}\right)+\left[\left(\frac{2n}{n}\right)^3-\frac{2n}{n}\right]\left(\frac{2}{n}\right)$$

$$\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3-\frac{2i}{n}\right]\left(\frac{2}{n}\right) \quad \text{or} \quad \boxed{\frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3-\frac{2i}{n}\right]}$$

17) Use properties of summation to evaluate the sum.

$$\sum_{i=1}^{20}(i-1)^2 = \sum_{i=1}^{20} i^2-2i+1$$

$$= \sum_{i=1}^{20} i^2 - 2\sum_{i=1}^{20} i + \sum_{i=1}^{20} 1$$

$$= \frac{n(n+1)(2n+1)}{6} - 2\cdot\frac{n(n+1)}{2} + n$$

$$= \frac{20(21)(41)}{6} - \cancel{2}\cdot\frac{(20)(21)}{\cancel{2}} + 20$$

$$= 2870 - 420 + 20$$

$$= \boxed{2470}$$

Find $\lim_{n\to\infty} s(n)$

*Compare degrees between numerator and denominator, much like finding horizontal asymptotes

31) $s(n) = \frac{81}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$

$= \lim_{n\to\infty} \frac{81}{n^4}\left[\frac{n^2(n^2+2n+1)}{4}\right]$

$= \lim_{n\to\infty} \frac{81}{n^4}\left[\frac{n^4+2n^3+n^2}{4}\right]$

$= \lim_{n\to\infty} \frac{81n^4+162n^3+81n^2}{4n^4}$

$= \boxed{\frac{81}{4}}$

33) $s(n) = \frac{18}{n^2}\left[\frac{n(n+1)}{2}\right]$

$= \lim_{n\to\infty} \frac{18(n^2+n)}{2n^2}$

$= \lim_{n\to\infty} \frac{18n^2+18n}{2n^2} = \frac{18}{2} = \boxed{9}$

Use summation formulas to find $\lim_{n\to\infty} s(n)$

Formulas:

1) $\sum 1 = n$
2) $\sum i = \frac{n^2}{2}+\frac{n}{2}$
3) $\sum i^2 = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$
4) $\sum i^3 = \frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$

39) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{16i}{n^2}$

*pull coefficients to the front. This will help with the formula substitution step.

*Reminder: $n$ is a constant

$= \lim_{n\to\infty} \frac{16}{n^2} \boxed{\sum_{i=1}^{n} i}$

$= \lim_{n\to\infty} \frac{16}{n^2}\left[\frac{n^2}{2}+\frac{n}{2}\right]$

$= \lim_{n\to\infty} \frac{16n^2}{2n^2} + \cancel{\frac{16n}{2n^2}}$

$= \frac{16}{2} = \boxed{8}$

41) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(i-1)^2$

$= \frac{1}{n^3}\left[\lim_{n\to\infty} \sum_{i=1}^{n} i^2 - 2i + 1\right]$

$= \frac{1}{n^3}\lim_{n\to\infty}\left[\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right]$

$= \frac{1}{n^3}\lim_{n\to\infty}\left[\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} - 2\left[\frac{n^2}{2} + \frac{n}{2}\right] + n\right]$

$= \lim_{n\to\infty}\left[\frac{n^3}{3n^3} + \frac{n^2}{2n^3} + \frac{n}{6n^3} - \frac{2n^2}{2n^3} - \frac{2n}{2n^3} + \frac{n}{n^3}\right]$

$= \boxed{\frac{1}{3}}$

43) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)\left(\frac{2}{n}\right)$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n} + \frac{2i}{n^2}\right)$

$= \lim_{n\to\infty}\left[\frac{2}{n}\sum_{i=1}^{n} 1 + \frac{2}{n^2}\sum_{i=1}^{n} i\right]$

$= \lim_{n\to\infty}\left[\frac{2}{n}(n) + \frac{2}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right)\right]$

$= \lim_{n\to\infty} \frac{2n}{n} + \frac{2n^2}{2n^2} + \frac{2\cancel{n}}{\cancel{2}n^2}$

$= 2 + \frac{2}{2}$

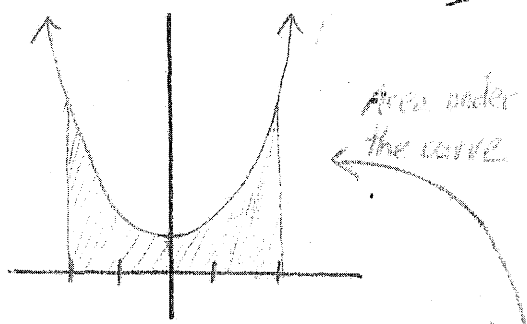
$= 2 + 1$

$= \boxed{3}$

# 4.2b – Riemann Sums

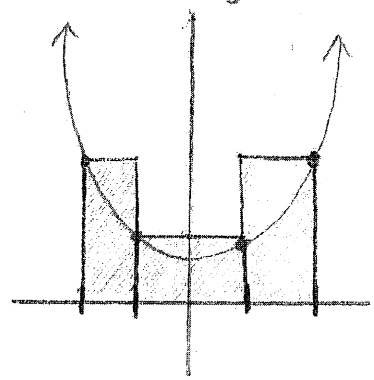
Riemann Sums – Using rectangles to estimate area of region.
(Area under a curve)

Consider the function
$f(x) = x^2 + 1 \quad [-2, 2]$




Suppose we want to estimate the area of the shaded region using a given number of rectangles.

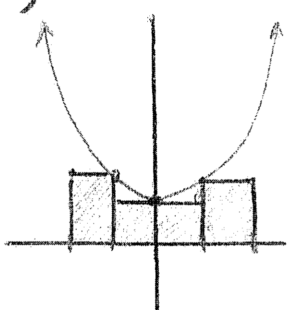
## 1) Upper rectangles or Circumscribed rectangles



* Using these rectangles will provide an overestimation of the area under the curve.

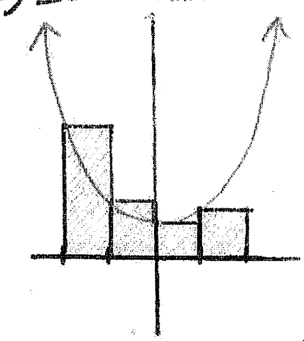
*Notice that one corner of each rectangle is on the graph. This ensures that the height of the rectangle is the same as the value of the function at the point where they connect.

## 2) Lower or Inscribed rectangles



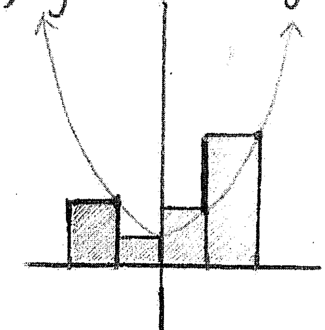
These rectangles provide an underestimation of the area under the curve

## 3) Left-handed Rectangles



The left corner of each rectangle attaches to the graph.

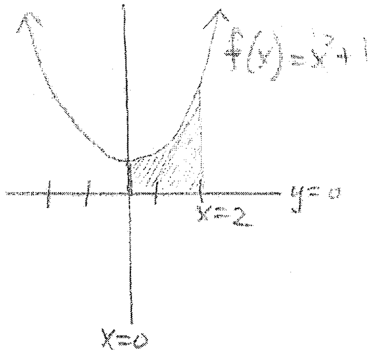
## 4) Right-handed Rectangles



The right corner of each rectangle attaches to the graph.

# 4.2b (continued)

**Ex. 1** Use 4 rectangles to estimate upper and lower sums for the area bounded by $x=0$, $y=0$, $x=2$, and $y=x^2+1$




Step 1: Determine width of each rectangle.

$$\boxed{\text{width} = \frac{b-a}{n}}$$

$a$ = left endpoint
$b$ = right endpoint
$n$ = number of rectangles

Step 2: Draw the graph. Section off each interval.

Step 3: Find sum of areas of appropriate rectangles.

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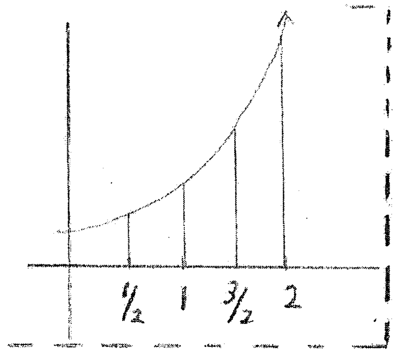
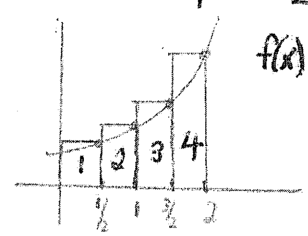



Since width $= \dfrac{b-a}{n}$, $\quad b=2$, $a=0$, $n=4$ $\quad$ width $= \dfrac{2-0}{4} = \dfrac{1}{2}$




a) Find upper sum

Area = width × height

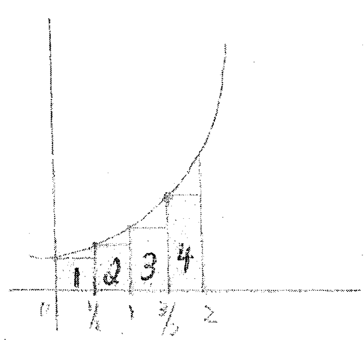
Rectangle #1: Area $= (1/2) \cdot f(1/2)$

$f(0.5) = (0.5)^2+1 = 1.25$
$f(1) = 1^2+1 = 2$
$f(1.5) = 1.5^2+1 = 3.25$
$f(2) = 2^2+1 = 5$

$$S(4) = \underbrace{\left(\tfrac{1}{2}\right)\left(f(1/2)\right)}_{\text{Area of Rectangle \#1}} + \underbrace{\tfrac{1}{2}f(1)}_{\text{Area of Rectangle \#2}} + \underbrace{\tfrac{1}{2}f(3/2)}_{\text{Area of Rectangle \#3}} + \underbrace{\tfrac{1}{2}f(2)}_{\text{Area of Rectangle \#4}}$$

$$S(4) = 0.625 + 1 + 1.625 + 2.5 = \boxed{5.75 \text{ or } 23/4}$$

b) Find lower sum




$$s(4) = \tfrac{1}{2}f(0) + \tfrac{1}{2}f(0.5) + \tfrac{1}{2}f(1) + \tfrac{1}{2}f(1.5)$$
$$= 0.5 + 0.625 + 1 + 1.625$$
$$= \boxed{3.75 \text{ or } 15/4}$$

$f(0) = 0^2+1 = 1$

* Good approximation for the actual area is the average between upper and lower sum: $\dfrac{5.75+3.75}{2} = \boxed{4.75}$

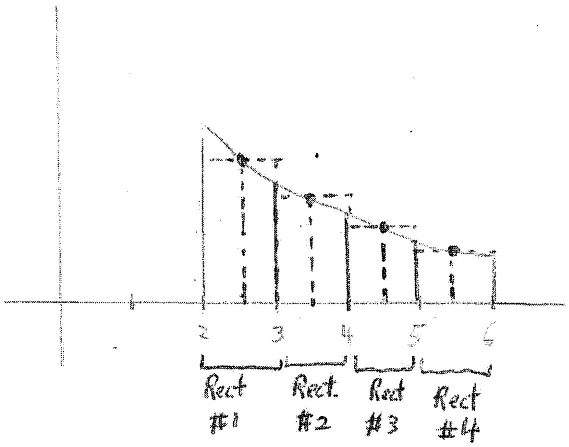
4.2b (continued)

Midpoint Rule: Similar to upper/lower sum but use the midpoint of each rectangle to calculate rectangle's height.

$\boxed{\text{Ex. 2}}$ Estimate area under curve $f(x) = \frac{8}{x^2+1}$ from $[2,6]$

Use midpoint rule with 4 subintervals.

$$\text{width} = \frac{b-a}{n} = \frac{6-2}{4} = \frac{4}{4} = 1$$




$$\text{Midpoint Sum} = \underbrace{1 \cdot f(2.5)}_{\text{Area Rect \#1}} + \underbrace{1 f(3.5)}_{\text{Area Rect \#2}} + \underbrace{1 f(4.5)}_{\text{Area Rect \#3}} + \underbrace{f(5.5)}_{\text{Area Rect \#4}}$$

$$f(2.5) = \frac{8}{2.5^2+1} = 1.103 \qquad f(4.5) = \frac{8}{4.5^2+1} = 0.376$$

$$f(3.5) = \frac{8}{3.5^2+1} = 0.604 \qquad f(5.5) = \frac{8}{5.5^2+1} = 0.256$$

$$\text{Midpt Sum} = 1.103 + 0.604 + 0.376 + 0.256 = \boxed{2.34}$$

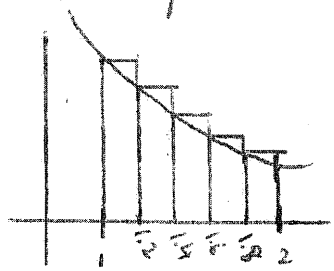
- Why would midpoint sum be a better approximation of area than upper or lower sum?

  This is because each rectangle has portions above and below the graph.

* Note: Midpoint sum is <u>not</u> the average between upper and lower sum!

4.2b Selected HW p.268-269 #23, 25, 27, 29, 63, 65

29) $y = \frac{1}{x}$



upper sum:

$f(1) = 1$

$f(1.2) = 1/1.2$

$f(1.4) = 1/1.4$

$f(1.6) = 1/1.6$

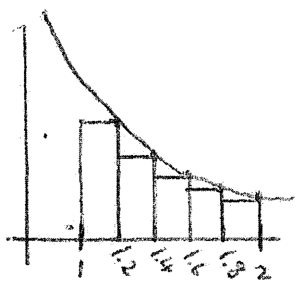
$f(1.8) = 1/1.8$

upper sum:

$\text{width} = \frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5} = 0.2$

$S = (0.2)f(1) + (0.2)f(1.2) + (0.2)f(1.4) + (0.2)f(1.6) + (0.2)f(1.8)$

$\approx \boxed{0.746}$



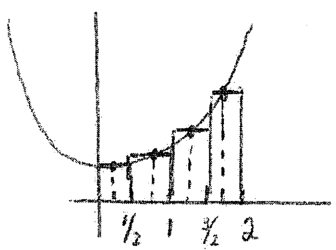
lower sum:

$s = (0.2)f(1.2) + (0.2)f(1.4) + (0.2)f(1.6) + (0.2)f(1.8) + (0.2)f(1)$

$\approx \boxed{0.646}$

63) Use Midpoint Rule: $n=4$

$f(x) = x^2 + 3 \quad [0,2]$




$\text{width} = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$

$f(1/4) = \frac{1}{16} + 3$

$f(3/4) = 9/16 + 3$

$f(5/4) = 25/16 + 3$

$f(7/4) = 49/16 + 3$

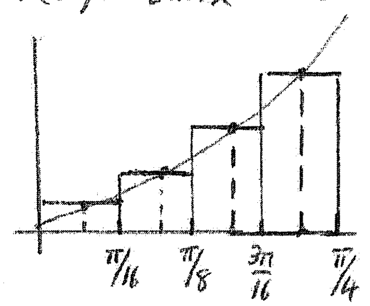
$\text{Area} \approx \frac{1}{2}f(1/4) + \frac{1}{2}f(3/4) + \frac{1}{2}f(5/4) + \frac{1}{2}f(7/4)$

$\approx \boxed{69/8}$

65) Midpt Rule: $n=4$

$f(x) = \tan x \quad 0 \le x \le \pi/4$

$\text{width} = \frac{\pi/4 - 0}{4} = \frac{\pi}{16}$




$f(\pi/32) = \tan(\pi/32)$

$f(3\pi/32) = \tan(3\pi/32)$

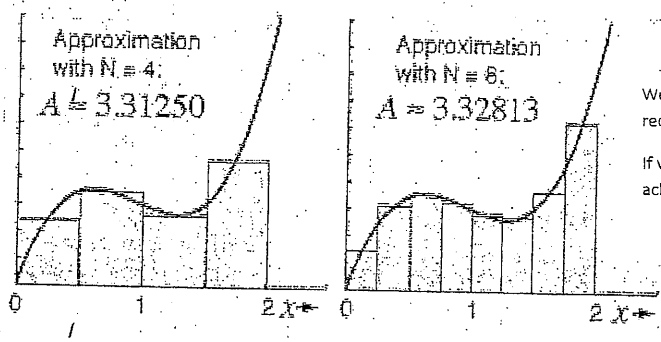
$f(5\pi/32) = \tan(5\pi/32)$

$f(7\pi/32) = \tan(7\pi/32)$

$\text{Area} \approx \frac{\pi}{16}f(\pi/32) + \frac{\pi}{16}f(3\pi/32) + \frac{\pi}{16}f\left(\frac{5\pi}{32}\right) + \frac{\pi}{16}f\left(\frac{7\pi}{32}\right)$

$\approx \boxed{0.345}$

# 4.2c Finding Exact Area using limits




We can continually improve the Area Approximation under the curve by increasing the number of rectangles : above (n = 4) and n = 8, ...n = 16...

If we let n go out to infinity, (using limits) we will have something better than an approximation, we will achieve the actual area under the curve.

$$\text{Exact Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{b-a}{n}\right) \cdot f\left(a + \left(\frac{b-a}{n}\right) i\right)\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (\text{width}) \cdot f(\underline{\text{left endpoint}} + \underline{\text{width} \cdot i})$$

Memorize → " width f left plus width times $i$ "

**Ex. 1** Find exact area between $f(x) = 4 - x^2$ and x-axis from $[-2, 2]$

$$\text{width} = \frac{b-a}{n} = \frac{2--2}{n} = \frac{4}{n}$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \cdot f\left(-2 + \frac{4}{n} i\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \cdot \left[4 - \left(-2 + \frac{4}{n} i\right)^2\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \cdot \left[4 - \left(4 - \frac{8i}{n} - \frac{8i}{n} + \frac{16}{n^2} i^2\right)\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \cdot \left[\cancel{4} - \cancel{4} + \frac{16i}{n} - \frac{16}{n^2} i^2\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \cdot \left[\frac{16i}{n} - \frac{16}{n^2} i^2\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{n^2} \sum_{i=1}^{n} i - \frac{64}{n^3} \sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right) - \frac{64}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{64n^2}{2n^2} + \cancel{\frac{64n}{2n^2}} - \frac{64n^3}{3n^3} - \cancel{\frac{64n^2}{2n^3}} - \cancel{\frac{64n}{6n^3}}\right]$$

$$= \frac{64}{2} - \frac{64}{3}$$

$$= 32 - \frac{64}{3}$$

$$= \boxed{\frac{32}{3}}$$

4.2c/4.6 Selected HW p. 269 #47-53 odd

p. 314 #1, 5, 9, 13, 17

Use limit process to find area of the region

49) $y = x^2 + 2$ $[0, 1]$ width $= \frac{b-a}{n} = \frac{1-0}{n}$ $\lim_{n\to\infty} \sum_{i=1}^{n} \text{width} \cdot f(\text{left} + \text{width} \cdot i)$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f\left(0 + \frac{1}{n} i\right)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{1}{n} i\right) = \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 + 2\right] = \sum_{i=1}^{n} \frac{i^2}{n^3} + \frac{2}{n}$$

$$\lim_{n\to\infty} \left[\frac{1}{n^3} \sum_{i=1}^{n} i^2 + \frac{2}{n} \sum_{i=1}^{n} 1\right] = \lim_{n\to\infty} \left[\frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) + \frac{2}{n}(n)\right]$$

$$\lim_{n\to\infty} \frac{n^3}{3n^3} + \cancel{\frac{n^2}{2n^3}} + \cancel{\frac{n}{6n^3}} + \frac{2n}{n} = \frac{1}{3} + 2 = \boxed{\frac{7}{3}}$$

51) $y = 16 - x^2$ $[1, 3]$ width $= \frac{3-1}{n} = \frac{2}{n}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} f\left(1 + \frac{2}{n} i\right) = \sum_{i=1}^{n} \frac{2}{n}\left[16 - \left(1 + \frac{2}{n} i\right)^2\right] = \sum_{i=1}^{n} \frac{2}{n}\left[16 - \left(1 + \frac{2}{n} i\right)\left(1 + \frac{2}{n} i\right)\right]$$

$$= \sum_{i=1}^{n} \frac{2}{n}\left[16 - \left(1 + \frac{4i}{n} + \frac{4}{n^2} i^2\right)\right] = \sum_{i=1}^{n} \frac{2}{n}\left[15 - \frac{4i}{n} - \frac{4i^2}{n^2}\right]$$

$$\lim_{n\to\infty} \left[\frac{30}{n} \sum_{i=1}^{n} 1 - \frac{8}{n^2} \sum_{i=1}^{n} i - \frac{8}{n^3} \sum_{i=1}^{n} i^2\right]$$

$$\lim_{n\to\infty} \left[\frac{30}{n}(n) - \frac{8}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right) - \frac{8}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\right]$$

$$\lim_{n\to\infty} \left[\frac{30n}{n} - \frac{8n^2}{2n^2} - \cancel{\frac{8n}{2n^2}} - \frac{8n^3}{3n^3} - \cancel{\frac{8n^2}{2n^3}} - \cancel{\frac{8n}{6n^3}}\right]$$

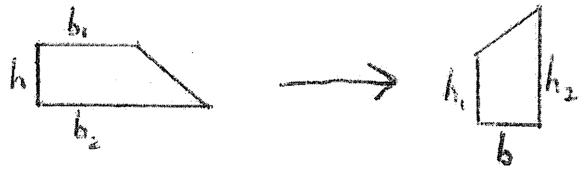
$$30 - 4 - \frac{8}{3} = \boxed{\frac{70}{3}}$$

# 4.6 Trapezoids

Trapezoidal Rule: Approximate Area of region using areas of trapezoids.

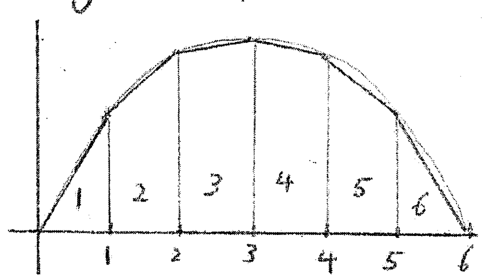
* Better approximation than inscribed, circumscribed, or midpoint rectangles.

<u>Review</u>: Area of Trapezoid $= \frac{1}{2}h(b_1+b_2)$ or $\frac{h}{2}(b_1+b_2)$




$\boxed{\text{Area} = \frac{b}{2}(h_1+h_2)}$ or $\frac{1}{2}w(h_1+h_2)$

$\boxed{\text{Ex. 1}}$ Estimate area bounded by $f(x) = 6x - x^2$ and the x-axis using 6 trapezoids.




Set $6x - x^2 = 0$ to find bounds for graph

$x(6-x) = 0 \quad \boxed{x=0, x=6}$

$\text{width} = \frac{b-a}{n} = \frac{6-0}{6} = \frac{6}{6} = \boxed{1}$

$f(0) = 6(0) - 0^2 = 0$

$f(1) = 6(1) - 1^2 = 5$

$f(2) = 6(2) - 2^2 = 8$

$f(3) = 6(3) - 3^2 = 9$

$f(4) = 6(4) - 4^2 = 8$

$f(5) = 6(5) - 5^2 = 5$

$f(6) = 6(6) - 6^2 = 0$

$\text{Area}_1 = \frac{1}{2}\left(f(0) + f(1)\right) = \frac{1}{2}(0+5) = 5/2$

$\text{Area}_2 = \frac{1}{2}\left(f(1) + f(2)\right) = \frac{1}{2}(13) = 13/2$

$\text{Area}_3 = \frac{1}{2}\left(f(2) + f(3)\right) = \frac{1}{2}(17) = 17/2$

$\text{Area}_4 = \frac{1}{2}\left(f(3) + f(4)\right) = \frac{1}{2}(17) = 17/2$

$\text{Area}_5 = \frac{1}{2}\left(f(4) + f(5)\right) = \frac{1}{2}(13) = 13/2$

$\text{Area}_6 = \frac{1}{2}\left(f(5) + f(6)\right) = \frac{1}{2}(5) = 5/2$

$\text{Area} = \frac{70}{2} = \boxed{35}$

* Integral Notation

$$\int_0^6 (6x - x^2)\,dx$$

6 ← upper bound (b)

$(6x-x^2)dx$ ← function

0 ← lower bound (a)

or

$$A = \frac{1}{2}\left[f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)\right] = \frac{1}{2}(70) = \boxed{35}$$

Pascal's Triangle

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$(a+b)^3 \longrightarrow 1 \quad 3 \quad 3 \quad 1$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

53) $y = 64 - x^3 \qquad [1, 4] \qquad \text{width} = \frac{4-1}{n} = \frac{3}{n}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} f\left(1 + \frac{3}{n}i\right)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[64 - \left(1 + \frac{3}{n}i\right)^3\right] = \sum_{i=1}^{n} \frac{3}{n}\left[64 - \left(1 + 3\left(\frac{3i}{n}\right) + 3\left(\frac{3}{n}i\right)^2 + \left(\frac{3}{n}i\right)^3\right)\right]$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[63 - \frac{9i}{n} - \frac{27}{n^2}i^2 - \frac{27}{n^3}i^3\right]$$

$$\lim_{n\to\infty}\left[\frac{189}{n}\sum_{i=1}^{n} 1 - \frac{27}{n^2}\sum_{i=1}^{n} i - \frac{81}{n^3}\sum_{i=1}^{n} i^2 - \frac{81}{n^4}\sum_{i=1}^{n} i^3\right]$$

$$\lim_{n\to\infty}\left[\frac{189}{n}(n) - \frac{27}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right) - \frac{81}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) - \frac{81}{n^4}\left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right)\right]$$

exponent degree should never be greater in the numerator.

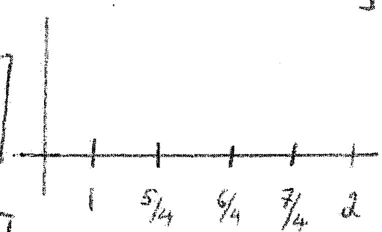
$$\lim_{n\to\infty}\left[\frac{189n}{n} - \frac{27n^2}{2n^2} - \cancel{\frac{27n}{2n^2}} - \frac{81n^3}{3n^3} - \cancel{\frac{81n^2}{2n^3}} - \cancel{\frac{81n}{6n^3}} - \frac{81n^4}{4n^4} - \cancel{\frac{81n^3}{2n^4}} - \cancel{\frac{81n^2}{4n^4}}\right]$$

$$= 189 - \frac{27}{2} - \frac{81}{3} - \frac{81}{4} = \boxed{\frac{513}{4} = 128.25}$$

4.6 Trapezoid Rule: p.314 #1, 5, 9, 13, 17

Approximate the definite integral using Trapezoidal Rule with $n=4$

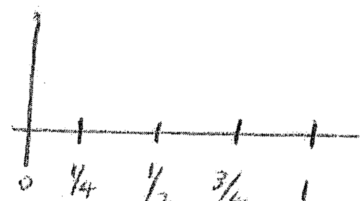
9) $\int_1^2 \frac{1}{(x+1)^2}\,dx$ width $= \frac{2-1}{4} = \frac{1}{4}$ $\frac{w}{2}\left[h_1 + 2h_2 + 2h_3 + \ldots + h_n\right]$




$A \approx \frac{1}{4}\left(\frac{1}{2}\right)\left[f(1) + 2f(5/4) + 2f(6/4) + 2f(7/4) + f(2)\right]$

$= \frac{1}{8}\left[\frac{1}{4} + \frac{32}{81} + \frac{8}{25} + \frac{32}{121} + \frac{1}{9}\right] \approx \boxed{0.1676}$

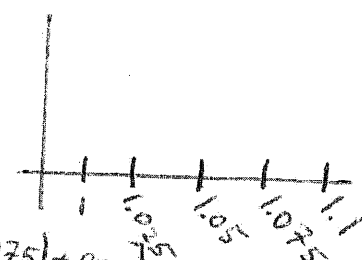
13) $\int_0^1 \sqrt{x}\,\sqrt{1-x}\,dx$ width $= \frac{1-0}{4} = \frac{1}{4}$




$A = \frac{1}{4}\left(\frac{1}{2}\right)\left[f(0) + 2f(1/4) + 2f(1/2) + 2f(3/4) + f(1)\right]$

$\approx \boxed{0.342}$

17) $\int_1^{1.1} \sin x^2\,dx$ width $= \frac{1.1-1}{4} = \frac{0.1}{4} = \frac{1}{40}$




$A = \frac{1}{40}\left(\frac{1}{2}\right)\left[f(1) + 2f(1.025) + 2f(1.05) + 2f(1.075) + f(1.1)\right]$

$A = \boxed{0.089}$