Section 4.2 Area

1. $\displaystyle\sum_{i=1}^{6}(3i + 2) = 3\sum_{i=1}^{6} i + \sum_{i=1}^{6} 2 = 3(1 + 2 + 3 + 4 + 5 + 6) + 12 = 75$

2. $\displaystyle\sum_{k=3}^{9}\left(k^2 + 1\right) = \left(3^2 + 1\right) + \left(4^2 + 1\right) + \dots + \left(9^2 + 1\right) = 287$

3. $\displaystyle\sum_{k=0}^{4}\frac{1}{k^2 + 1} = 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} = \frac{158}{85}$

4. $\displaystyle\sum_{j=4}^{6}\frac{3}{j} = \frac{3}{4} + \frac{3}{5} + \frac{3}{6} = \frac{37}{20}$

5. $\displaystyle\sum_{k=1}^{4} c = c + c + c + c = 4c$

6. $\displaystyle\sum_{i=1}^{4}\left[(i - 1)^2 + (i + 1)^3\right] = (0 + 8) + (1 + 27) + (4 + 64) + (9 + 125) = 238$

7. $\displaystyle\sum_{i=1}^{11}\frac{1}{5i}$

8. $\displaystyle\sum_{i=1}^{14}\frac{9}{1 + i}$

9. $\displaystyle\sum_{j=1}^{6}\left[7\left(\frac{j}{6}\right) + 5\right]$

10. $\displaystyle\sum_{j=1}^{4}\left[1 - \left(\frac{j}{4}\right)^2\right]$

11. $\displaystyle\frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right)\right]$

12. $\displaystyle\frac{3}{n}\sum_{i=1}^{n}\left[2\left(1 + \frac{3i}{n}\right)^2\right]$

13. $\displaystyle\sum_{i=1}^{12} 7 = 7(12) = 84$

14. $\displaystyle\sum_{i=1}^{30}(-18) = (-18)(30) = -540$

15. $\displaystyle\sum_{i=1}^{24} 4i = 4\sum_{i=1}^{24} i = 4\left[\frac{24(25)}{2}\right] = 1200$

16. $\displaystyle\sum_{i=1}^{16}(5i - 4) = 5\sum_{i=1}^{16} i - 4(16) = 5\left[\frac{16(17)}{2}\right] - 64 = 616$

17. $\displaystyle\sum_{i=1}^{20}(i - 1)^2 = \sum_{i=1}^{19} i^2 = \left[\frac{19(20)(39)}{6}\right] = 2470$

18. $\displaystyle\sum_{i=1}^{10}\left(i^2 - 1\right) = \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} 1 = \left[\frac{10(11)(21)}{6}\right] - 10 = 375$

19. $\displaystyle\sum_{i=1}^{15} i(i - 1)^2 = \sum_{i=1}^{15} i^3 - 2\sum_{i=1}^{15} i^2 + \sum_{i=1}^{15} i$

$$= \frac{15^2(16)^2}{4} - 2\frac{15(16)(31)}{6} + \frac{15(16)}{2}$$

$$= 14{,}400 - 2480 + 120 = 12{,}040$$

20. $\displaystyle\sum_{i=1}^{25}\left(i^3 - 2i\right) = \sum_{i=1}^{25} i^3 - 2\sum_{i=1}^{25} i$

$$= \frac{(25)^2(26)^2}{4} - 2\frac{25(26)}{2}$$

$$= 105{,}625 - 650$$

$$= 104{,}975$$

21. $\displaystyle\sum_{i=1}^{n}\frac{2i+1}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n}(2i+1) = \frac{1}{n^2}\left[2\frac{n(n+1)}{2} + n\right] = \frac{n+2}{n} = 1 + \frac{2}{n} = S(n)$

$$S(10) = \frac{12}{10} = 1.2$$
$$S(100) = 1.02$$
$$S(1000) = 1.002$$
$$S(10{,}000) = 1.0002$$

22. $\displaystyle\sum_{j=1}^{n}\frac{7j+4}{n^2} = \frac{1}{n^2}\sum_{j=1}^{n}(7j+4)$

$$= \frac{1}{n^2}\left[7\frac{n(n+1)}{2} + 4n\right]$$
$$= \frac{7n^2+7n}{2n^2} + \frac{4n}{n^2} = \frac{7n+15}{2n} = S(n)$$
$$S(10) = \tfrac{17}{4} = 4.25$$
$$S(100) = 3.575$$
$$S(1000) = 3.5075$$
$$S(10{,}000) = 3.50075$$

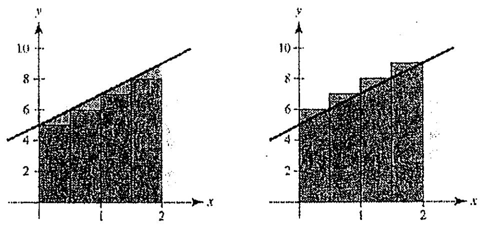
23. $\displaystyle\sum_{k=1}^{n}\frac{6k(k-1)}{n^3} = \frac{6}{n^3}\sum_{k=1}^{n}\left(k^2-k\right) = \frac{6}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]$

$$= \frac{6}{n^2}\left[\frac{2n^2+3n+1-3n-3}{6}\right] = \frac{1}{n^2}\left[2n^2-2\right] = 2 - \frac{2}{n^2} = S(n)$$
$$S(10) = 1.98$$
$$S(100) = 1.9998$$
$$S(1000) = 1.999998$$
$$S(10{,}000) = 1.99999998$$

24. $\displaystyle\sum_{i=1}^{n}\frac{2i^3-3i}{n^4} = \frac{1}{n^4}\sum_{i=1}^{n}\left(2i^3-3i\right)$

$$= \frac{1}{n^4}\left[2\frac{n^2(n+1)^2}{4} - 3\frac{n(n+1)}{2}\right]$$
$$= \frac{(n+1)^2}{2n^2} - \frac{3(n+1)}{2n^3} = \frac{n^3+2n^2-2n-3}{2n^3} = S(n)$$
$$S(10) = 0.5885$$
$$S(100) = 0.5098985$$
$$S(1000) = 0.5009989985$$
$$S(10{,}000) = 0.50009999$$

25.



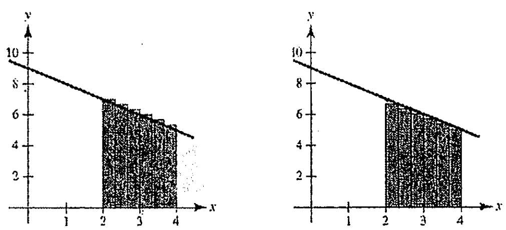
$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

Left endpoints: Area $\approx \frac{1}{2}[5 + 6 + 7 + 8] = \frac{26}{2} = 13$

Right endpoints: Area $\approx \frac{1}{2}[6 + 7 + 8 + 9] = \frac{30}{2} = 15$

$13 < \text{Area} < 15$

26.



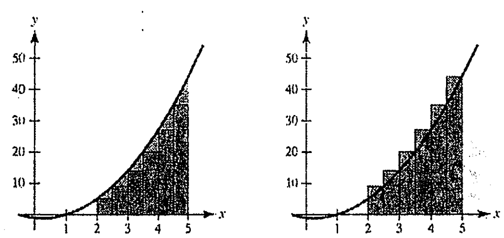
$$\Delta x = \frac{4-2}{6} = \frac{1}{3}$$

Left endpoints: Area $\approx \frac{1}{3}\left[7 + \frac{20}{3} + \frac{19}{3} + 6 + \frac{17}{3} + \frac{16}{3}\right] = \frac{37}{3} \approx 12.333$

Right endpoints: Area $\approx \frac{1}{3}\left[\frac{20}{3} + \frac{19}{3} + 6 + \frac{17}{3} + \frac{16}{3} + \frac{15}{3}\right] = \frac{35}{3} \approx 11.667$

$\frac{35}{3} < \text{Area} < \frac{37}{3}$

27.



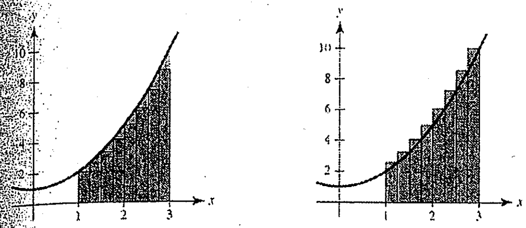
$$\Delta x = \frac{5-2}{6} = \frac{1}{2}$$

Left endpoints: Area $\approx \frac{1}{2}[5 + 9 + 14 + 20 + 27 + 35] = 55$

Right endpoints: Area $\approx \frac{1}{2}[9 + 14 + 20 + 27 + 35 + 44] = \frac{149}{2} = 74.5$

$55 < \text{Area} < 74.5$

28.



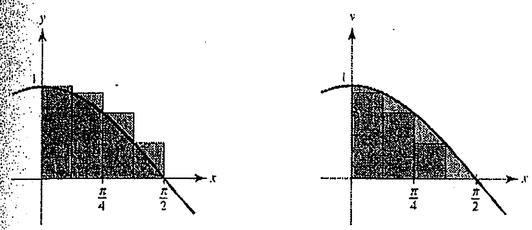
$\Delta x = \frac{3-1}{8} = \frac{1}{4}$

Left endpoints: Area $\approx \frac{1}{4}\left[2 + \frac{41}{16} + \frac{13}{4} + \frac{65}{16} + 5 + \frac{97}{16} + \frac{29}{4} + \frac{137}{16}\right] = \frac{155}{16} = 9.6875$

Right endpoint: Area $\approx \frac{1}{4}\left[\frac{41}{16} + \frac{13}{4} + \frac{65}{16} + 5 + \frac{97}{16} + \frac{29}{4} + \frac{137}{16} + 10\right] = 11.6875$

$9.6875 < \text{Area} < 11.6875$

29.



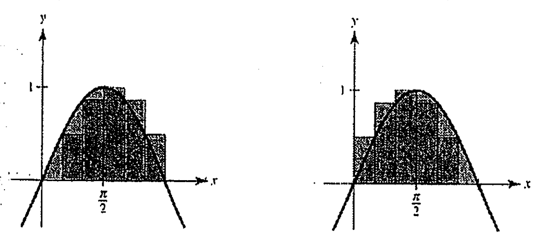
$\Delta x = \frac{\frac{\pi}{2} - 0}{4} = \frac{\pi}{8}$

Left endpoints: Area $\approx \frac{\pi}{8}\left[\cos(0) + \cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{8}\right)\right] \approx 1.1835$

Right endpoints: Area $\approx \frac{\pi}{8}\left[\cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{8}\right) + \cos\left(\frac{\pi}{2}\right)\right] \approx 0.7908$

$0.7908 < \text{Area} < 1.1835$

30.



$\Delta x = \frac{\pi - 0}{6} = \frac{\pi}{6}$

Left endpoints: Area $\approx \frac{\pi}{6}\left[\sin 0 + \sin\frac{\pi}{6} + \sin\frac{\pi}{3} + \sin\frac{\pi}{2} + \sin\frac{2\pi}{3} + \sin\frac{5\pi}{6}\right] \approx 1.9541$

Right endpoints: Area $\approx \frac{\pi}{6}\left[\sin\frac{\pi}{6} + \sin\frac{\pi}{3} + \sin\frac{\pi}{2} + \sin\frac{2\pi}{3} + \sin\frac{5\pi}{6} + \sin\pi\right] \approx 1.9541$

By symmetry, the answers are the same. The exact area (2) is larger.

31. $S = \left[3 + 4 + \frac{9}{2} + 5\right](1) = \frac{33}{2} = 16.5$

$s = \left[1 + 3 + 4 + \frac{9}{2}\right](1) = \frac{25}{2} = 12.5$

32. $S = [5 + 5 + 4 + 2](1) = 16$

$s = [4 + 4 + 2 + 0](1) = 10$

33. $S(4) = \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) + \sqrt{1}\left(\frac{1}{4}\right) = \frac{1 + \sqrt{2} + \sqrt{3} + 2}{8} \approx 0.768$

$s(4) = 0\left(\frac{1}{4}\right) + \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) = \frac{1 + \sqrt{2} + \sqrt{3}}{8} \approx 0.518$

34. $S(8) = \left(\sqrt{\frac{1}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{3}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{1} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{5}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{3}{2}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{7}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{2} + 2\right)\frac{1}{4}$

$= \frac{1}{4}\left(16 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{5}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{7}}{2} + \sqrt{2}\right) \approx 6.038$

$s(8) = (0 + 2)\frac{1}{4} + \left(\sqrt{\frac{1}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2\right)\frac{1}{4} + \dots + \left(\sqrt{\frac{7}{4}} + 2\right)\frac{1}{4} \approx 5.685$

35. $S(5) = 1\left(\frac{1}{5}\right) + \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \approx 0.746$

$s(5) = \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646$

36. $S(5) = 1\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right)$

$= \frac{1}{5}\left[1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5}\right] \approx 0.859$

$s(5) = \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + 0 \approx 0.659$

37. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{24i}{n^2}\right) = \lim_{n\to\infty} \frac{24}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{24}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\left[12\left(\frac{n^2 + n}{n^2}\right)\right] = 12 \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 12$

38. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \lim_{n\to\infty} \frac{9}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{9}{n^2}\left[\frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \frac{9}{2}\left(\frac{n+1}{n}\right) = \frac{9}{2}$

39. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(i - 1)^2 = \lim_{n\to\infty} \frac{1}{n^3}\sum_{i=1}^{n-1} i^2 = \lim_{n\to\infty} \frac{1}{n^3}\left[\frac{(n-1)(n)(2n-1)}{6}\right]$

$= \lim_{n\to\infty} \frac{1}{6}\left[\frac{2n^3 - 3n^2 + n}{n^3}\right] = \lim_{n\to\infty}\left[\frac{1}{6}\left(\frac{2 - (3/n) + (1/n^2)}{1}\right)\right] = \frac{1}{3}$

40. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^2\left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2}{n^3}\sum_{i=1}^{n} (n + 2i)^2$

$= \lim_{n\to\infty} \frac{2}{n^3}\left[\sum_{i=1}^{n} n^2 + 4n\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} i^2\right]$

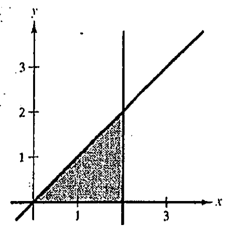
$= \lim_{n\to\infty} \frac{2}{n^3}\left[n^3 + (4n)\left(\frac{n(n+1)}{2}\right) + \frac{4(n)(n+1)(2n+1)}{6}\right]$

$= 2 \lim_{n\to\infty}\left[1 + 2 + \frac{2}{n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^2}\right] = 2\left(1 + 2 + \frac{4}{3}\right) = \frac{26}{3}$

41. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)\left(\frac{2}{n}\right) = 2\lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n}1+\frac{1}{n}\sum_{i=1}^{n}i\right] = 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}\left(\frac{n(n+1)}{2}\right)\right] = 2\lim_{n\to\infty}\left[1+\frac{n^2+n}{2n^2}\right] = 2\left(1+\frac{1}{2}\right) = 3$

42. $$\lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{3i}{n}\right)^3\left(\frac{3}{n}\right) = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\frac{2n+3i}{n}\right]^3$$
$$= \lim_{n\to\infty}\frac{3}{n^4}\sum_{i=1}^{n}\left(8n^3+36n^2i+54ni^2+27i^3\right)$$
$$= \lim_{n\to\infty}\frac{3}{n^4}\left(8n^4+36n^2\frac{n(n+1)}{2}+54n\frac{n(n+1)(2n+1)}{6}+27\frac{n^2(n+1)^2}{4}\right)$$
$$= \lim_{n\to\infty}3\left(8+18\frac{(n+1)}{n}+\frac{9(n+1)(2n+1)}{n^2}+\frac{27}{4}\cdot\frac{(n+1)^2}{n^2}\right)$$
$$= 3\left(8+18+18+\frac{27}{4}\right) = \frac{609}{4} = 152.25$$

43. (a)



(b) $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

Endpoints: $0 < 1\left(\frac{2}{n}\right) < 2\left(\frac{2}{n}\right) < \dots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right) = 2$

(c) Because $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} f\left(\frac{2i-2}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

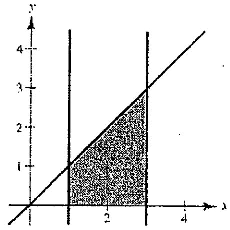
(e)

x	5	10	50	100
$s(n)$	1.6	1.8	1.96	1.98
$S(n)$	2.4	2.2	2.04	2.02

(f) $\lim_{n\to\infty}\sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n}(i-1) = \lim_{n\to\infty}\frac{4}{n^2}\left[\frac{n(n+1)}{2}-n\right] = \lim_{n\to\infty}\left[\frac{2(n+1)}{n}-\frac{4}{n}\right] = 2$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n}i = \lim_{n\to\infty}\left(\frac{4}{n^2}\right)\frac{n(n+1)}{2} = \lim_{n\to\infty}\frac{2(n+1)}{n} = 2$$

44. (a)



(b) $\Delta x = \frac{3-1}{n} = \frac{2}{n}$

Endpoints:

$$1 < 1 + \frac{2}{n} < 1 + \frac{4}{n} < \cdots < 1 + \frac{2n}{n} = 3$$

$$1 < 1 + 1\left(\frac{2}{n}\right) < 1 + 2\left(\frac{2}{n}\right) < \cdots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right)$$

(c) Because $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x = \sum_{i=1}^{n} f\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

x	5	10	50	100
$s(n)$	3.6	3.8	3.96	3.98
$S(n)$	4.4	4.2	4.04	4.02

(f) $$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\left(\frac{2}{n}\right)\left[n + \frac{2}{n}\left(\frac{n(n+1)}{2} - n\right)\right]$$

$$= \lim_{n\to\infty}\left[2 + \frac{2n+2}{n} - \frac{4}{n}\right] = \lim_{n\to\infty}\left[4 - \frac{2}{n}\right] = 4$$

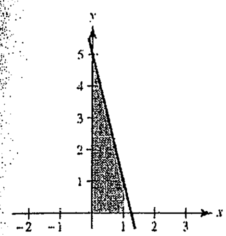
$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n}\left[n + \left(\frac{2}{n}\right)\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[2 + \frac{2(n+1)}{n}\right] = \lim_{n\to\infty}\left[4 + \frac{2}{n}\right] = 4$$

45. $y = -4x + 5$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[-4\left(\frac{i}{n}\right) + 5\right]\left(\frac{1}{n}\right)$$
$$= -\frac{4}{n^2}\sum_{i=1}^{n} i + 5$$
$$= -\frac{4}{n^2}\frac{n(n+1)}{2} + 5$$
$$= -2\left(1 + \frac{1}{n}\right) + 5$$

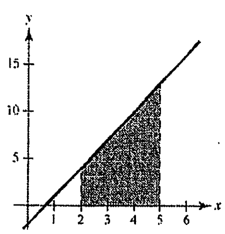
$$\text{Area} = \lim_{n\to\infty} s(n) = 3$$



46. $y = 3x - 2$ on $[2, 5]$. $\left(\textbf{Note: } \Delta x = \dfrac{5-2}{n} = \dfrac{3}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$
$$= \sum_{i=1}^{n}\left[3\left(2 + \frac{3i}{n}\right) - 2\right]\left(\frac{3}{n}\right)$$
$$= 18 + 3\left(\frac{3}{n}\right)^2\sum_{i=1}^{n} i - 6$$
$$= 12 + \frac{27}{n^2}\left(\frac{(n+1)n}{2}\right) = 12 + \frac{27}{2}\left(1 + \frac{1}{n}\right)$$

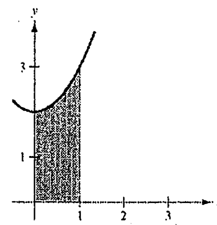
$$\text{Area} = \lim_{n\to\infty} S(n) = 12 + \frac{27}{2} = \frac{51}{2}$$



47. $y = x^2 + 2$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right)$$
$$= \sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2 + 2\right]\left(\frac{1}{n}\right)$$
$$= \left[\frac{1}{n^3}\sum_{i=1}^{n} i^2\right] + 2$$
$$= \frac{n(n+1)(2n+1)}{6n^3} + 2 = \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 2$$

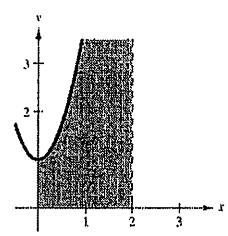
$$\text{Area} = \lim_{n\to\infty} S(n) = \frac{7}{3}$$



48. $y = 3x^2 + 1$ on $[0, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[3\left(\frac{2i}{n}\right)^2 + 1\right]\left(\frac{2}{n}\right)$$
$$= \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} 1$$
$$= \frac{24}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{2}{n}(n)$$
$$= \frac{4(n+1)(2n+1)}{n^2} + 2$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 8 + 2 = 10$$



49. $y = 25 - x^2$ on $[1, 4]$. $\left(\textbf{Note: } \Delta x = \dfrac{3}{n}\right)$

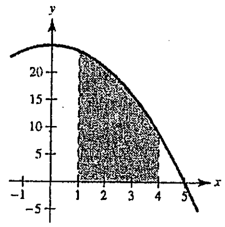
$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left[25 - \left(1 + \frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

$$= \frac{3}{n}\sum_{i=1}^{n}\left[24 - \frac{9i^2}{n^2} - \frac{6i}{n}\right]$$

$$= \frac{3}{n}\left[24n - \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{6}{n}\frac{n(n+1)}{2}\right]$$

$$= 72 - \frac{9}{2n^2}(n+1)(2n+1) - \frac{9}{n}(n+1)$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 72 - 9 - 9 = 54$$



50. $y = 4 - x^2$ on $[-2, 2]$. Find area of region over the interval $[0, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

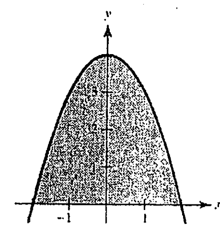
$$= \sum_{i=1}^{n}\left[4 - \left(\frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right)$$

$$= 8 - \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

$$= 8 - \frac{8n(n+1)(2n+1)}{6n^3} = 8 - \frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\frac{1}{2}\text{Area} = \lim_{n\to\infty} s(n) = 8 - \frac{8}{3} = \frac{16}{3}$$

$$\text{Area} = \frac{32}{3}$$



51. $y = 27 - x^3$ on $[1, 3]$. $\left(\textbf{Note: } \Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}\right)$

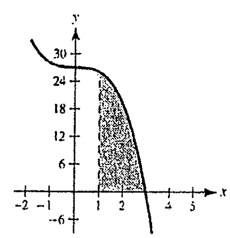
$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[27 - \left(1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right)$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[26 - \frac{8i^3}{n^3} - \frac{12i^2}{n^2} - \frac{6i}{n}\right]$$

$$= \frac{2}{n}\left[26n - \frac{8}{n^3}\frac{n^2(n+1)^2}{4} - \frac{12}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{6}{n}\frac{n(n+1)}{2}\right]$$

$$= 52 - \frac{4}{n^2}(n+1)^2 - \frac{4}{n^2}(n+1)(2n+1) - \frac{6n+1}{n}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 52 - 4 - 8 - 6 = 34$$



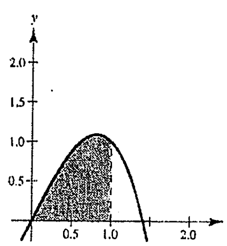
52. $y = 2x - x^3$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}\right)$

Because y both increases and decreases on $[0, 1]$, $T(n)$ is neither an upper nor lower sum.

$$T(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$$

$$= \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{n(n+1)}{n^2} - \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = 1 + \frac{1}{n} - \frac{1}{4} - \frac{2}{4n} - \frac{1}{4n^2}$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 1 - \frac{1}{4} = \frac{3}{4}$$



53. $y = x^2 - x^3$ on $[-1, 1]$. $\left(\textbf{Note: } \Delta x = \frac{1-(-1)}{n} = \frac{2}{n}\right)$

Because y both increases and decreases on $[-1, 1]$, $T(n)$ is neither an upper nor a lower sum.

$$T(n) = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right)$$

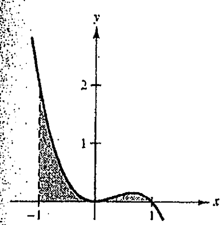
$$= \sum_{i=1}^{n}\left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right) = \frac{4}{n}\sum_{i=1}^{n}1 - \frac{20}{n^2}\sum_{i=1}^{n}i + \frac{32}{n^3}\sum_{i=1}^{n}i^2 - \frac{16}{n^4}\sum_{i=1}^{n}i^3$$

$$= \frac{4}{n}(n) - \frac{20}{n^2}\cdot\frac{n(n+1)}{2} + \frac{32}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4}$$

$$= 4 - 10\left(1 + \frac{1}{n}\right) + \frac{16}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$$



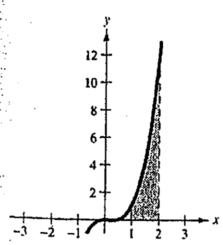
54. $y = 2x^3 - x^2$ on $[1, 2]$. $\left(\textbf{Note: } \Delta x = \frac{2-1}{n} = \frac{1}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\left(1 + \frac{i}{n}\right)^3 - \left(1 + \frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left(\frac{2i^3}{n^3} + \frac{5i^2}{n^2} + \frac{4i}{n} + 1\right)\left(\frac{1}{n}\right)$$

$$= \frac{2}{n^4}\cdot\frac{n^2(n+1)^2}{4} + \frac{5}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{4}{n^2}\cdot\frac{n(n+1)}{2} + 1$$

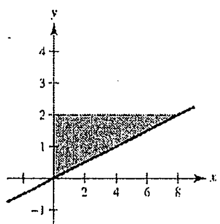
$$\text{Area} = \lim_{n\to\infty} s_n = \frac{1}{2} + \frac{5}{3} + 2 + 1 = \frac{31}{6}$$



55. $f(y) = 4y,\ 0 \le y \le 2$ $\left(\textbf{Note: } \Delta y = \frac{2-0}{n} = \frac{2}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f(m_i)\Delta y = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n} 4\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \frac{16}{n^2}\sum_{i=1}^{n} i = \left(\frac{16}{n^2}\right)\cdot\frac{n(n+1)}{2} = \frac{8(n+1)}{n} = 8 + \frac{8}{n}$$

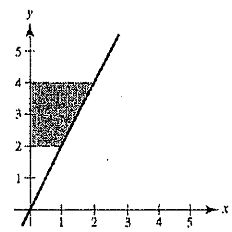
$$\text{Area} = \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(8 + \frac{8}{n}\right) = 8$$



56. $g(y) = \frac{1}{2}y,\ 2 \le y \le 4.$ $\left(\textbf{Note: } \Delta y = \frac{4-2}{n} = \frac{2}{n}\right)$

$$S(n) = \sum_{i=1}^{n} g\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n} \frac{1}{2}\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \frac{2}{n}\sum_{i=1}^{n}\left(1 + \frac{i}{n}\right) = \frac{2}{n}\left[n + \frac{1}{n}\frac{n(n+1)}{2}\right] = 2 + \frac{n+1}{n}$$

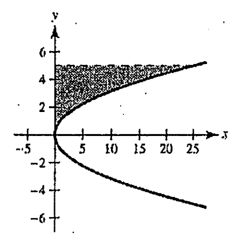
$$\text{Area} = \lim_{n\to\infty} S(n) = 2 + 1 = 3$$



57. $f(y) = y^2,\ 0 \le y \le 5$ $\left(\textbf{Note: } \Delta y = \frac{5-0}{n} = \frac{5}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(\frac{5i}{n}\right)\left(\frac{5}{n}\right) = \sum_{i=1}^{n}\left(\frac{5i}{n}\right)^2\left(\frac{5}{n}\right) = \frac{125}{n^3}\sum_{i=1}^{n} i^2 = \frac{125}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{125}{n^2}\left(\frac{2n^2+3n+1}{6}\right) = \frac{125}{3} + \frac{125}{2n} + \frac{125}{6n^2}$$

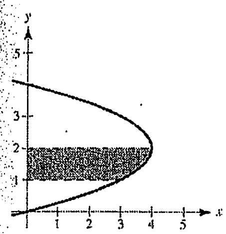
$$\text{Area } \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(\frac{125}{3} + \frac{125}{2n} + \frac{125}{6n^2}\right) = \frac{125}{3}$$



58. $f(y) = 4y - y^2,\ 1 \le y \le 2.\ \left(\textbf{Note: } \Delta y = \frac{2-1}{n} = \frac{1}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left[4\left(1 + \frac{i}{n}\right) - \left(1 + \frac{i}{n}\right)^2\right]$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(4 + \frac{4i}{n} - 1 - \frac{2i}{n} - \frac{i^2}{n^2}\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(3 + \frac{2i}{n} - \frac{i^2}{n^2}\right)$$
$$= \frac{1}{n}\left[3n + \frac{2}{n}\frac{n(n+1)}{2} - \frac{1}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$
$$= 3 + \frac{n+1}{n} - \frac{(n+1)(2n+1)}{6}$$

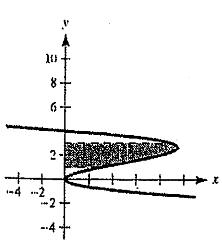
$$\text{Area} = \lim_{n\to\infty} S(n) = 3 + 1 - \frac{1}{3} = \frac{11}{3}$$



59. $g(y) = 4y^2 - y^3,\ 1 \le y \le 3.\ \left(\textbf{Note: } \Delta y = \frac{3-1}{n} = \frac{2}{n}\right)$

$$S(n) = \sum_{i=1}^{n} g\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$
$$= \sum_{i=1}^{n}\left[4\left(1 + \frac{2i}{n}\right)^2 - \left(1 + \frac{2i}{n}\right)^3\right]\frac{2}{n}$$
$$= \frac{2}{n}\sum_{i=1}^{n} 4\left[1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right] - \left[1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[3 + \frac{10i}{n} + \frac{4i^2}{n^2} - \frac{8i^3}{n^3}\right]$$
$$= \frac{2}{n}\left[3n + \frac{10}{n}\frac{n(n+1)}{2} + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2}\frac{n^2(n+1)^2}{4}\right]$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 6 + 10 + \frac{8}{3} - 4 = \frac{44}{3}$$



60. $h(y) = y^3 + 1, 1 \le y \le 2 \left(\textbf{Note: } \Delta y = \frac{1}{n}\right)$

$$S(n) = \sum_{i=1}^{n} h\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right)$$

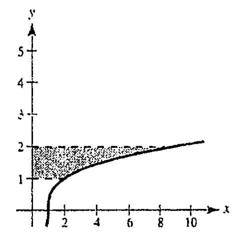
$$= \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^3 + 1\right]\frac{1}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(2 + \frac{i^3}{n^3} + \frac{3i^2}{n^2} + \frac{3i}{n}\right)$$

$$= \frac{1}{n}\left[2n + \frac{1}{n^3}\frac{n^2(n+1)^2}{4} + \frac{3}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\frac{3n(n+1)}{2n}\right]$$

$$= 2 + \frac{(n+1)^2}{n^2 4} + \frac{1}{2}\frac{(n+1)(2n+1)}{n^2} + \frac{3(n+1)}{2n}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 2 + \frac{1}{4} + 1 + \frac{3}{2} = \frac{19}{4}$$



61. $f(x) = x^2 + 3, 0 \le x \le 2, n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$\Delta x = \frac{1}{2}, c_1 = \frac{1}{4}, c_2 = \frac{3}{4}, c_3 = \frac{5}{4}, c_4 = \frac{7}{4}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4}\left[c_i^2 + 3\right]\left(\frac{1}{2}\right) = \frac{1}{2}\left[\left(\frac{1}{16} + 3\right) + \left(\frac{9}{16} + 3\right) + \left(\frac{25}{16} + 3\right) + \left(\frac{49}{16} + 3\right)\right] = \frac{69}{8}$$

62. $f(x) = x^2 + 4x, 0 \le x \le 4, n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$\Delta x = 1, c_1 = \frac{1}{2}, c_2 = \frac{3}{2}, c_3 = \frac{5}{2}, c_4 = \frac{7}{2}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}\left[c_i^2 + 4c_i\right](1) = \left[\left(\frac{1}{4} + 2\right) + \left(\frac{9}{4} + 6\right) + \left(\frac{25}{4} + 10\right) + \left(\frac{49}{4} + 14\right)\right] = 53$$

63. $f(x) = \tan x, 0 \le x \le \frac{\pi}{4}, n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$\Delta x = \frac{\pi}{16}, c_1 = \frac{\pi}{32}, c_2 = \frac{3\pi}{32}, c_3 = \frac{5\pi}{32}, c_4 = \frac{7\pi}{32}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4}(\tan c_i)\left(\frac{\pi}{16}\right) = \frac{\pi}{16}\left(\tan\frac{\pi}{32} + \tan\frac{3\pi}{32} + \tan\frac{5\pi}{32} + \tan\frac{7\pi}{32}\right) \approx 0.345$$

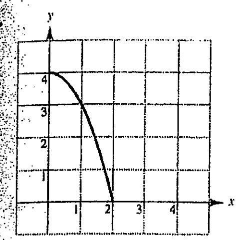
64. $f(x) = \cos x,\ 0 \le x \le \dfrac{\pi}{2},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i+1}}{2}$.

$$\Delta x = \frac{\pi}{8},\ c_1 = \frac{\pi}{16},\ c_2 = \frac{3\pi}{16},\ c_3 = \frac{5\pi}{16},\ c_4 = \frac{7\pi}{16}$$

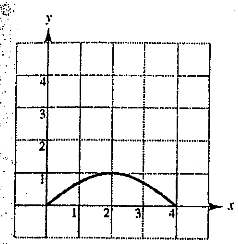
$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4} \cos(ci)\left(\frac{\pi}{8}\right) = \frac{\pi}{8}\left(\cos\frac{\pi}{16} + \cos\frac{3\pi}{16} + \cos\frac{5\pi}{16} + \cos\frac{7\pi}{16}\right) \approx 1.006$$

65.



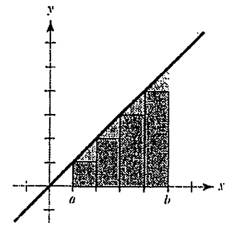
(b) $A \approx 6$ square units

66.

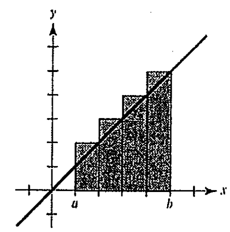


(a) $A \approx 3$ square units

67. You can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of these inscribed rectangles is the lower sum.



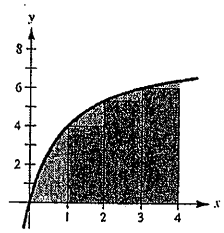
The sum of the areas of these circumscribed rectangles is the upper sum.



You can see that the rectangles do not contain all of the area in the first graph and the rectangles in the second graph cover more than the area of the region. The exact value of the area lies between these two sums.

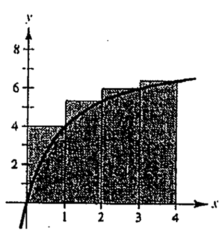
68. See the definition of area, page 260.

69. (a)



Lower sum: $s(4) = 0 + 4 + 5\frac{1}{3} + 6 = 15\frac{1}{3} = \frac{46}{3} \approx 15.333$

(b)



Upper sum: $S(4) = 4 + 5\frac{1}{3} + 6 + 6\frac{2}{5} = 21\frac{11}{15} = \frac{326}{15} \approx 21.733$

102. $\alpha^2 \int_0^1 f(x)\,dx = \alpha^2(1) = \alpha^2$

$-2\alpha \int_0^1 f(x)x\,dx = -2\alpha(\alpha) = -2\alpha^2$

$\int_0^1 f(x)x^2\,dx = \alpha^2$

Adding,

$\int_0^1 \left[\alpha^2 f(x) - 2\alpha\, xf(x) + x^2 f(x)\right] dx = 0$

$\int_0^1 f(x)(\alpha - x)^2\,dx = 0.$

Because $(\alpha - x)^2 \geq 0$, $f = 0$. So, there are no such functions.

Section 4.6 Numerical Integration

1. Exact: $\int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3} \approx 2.6667$

Trapezoidal: $\int_0^2 x^2\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{11}{4} = 2.7500$

Simpson's: $\int_0^2 x^2\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{8}{3} \approx 2.6667$

2. Exact: $\int_1^2 \left(\frac{x^2}{4} + 1\right) dx = \left[\frac{x^3}{12} + x\right]_1^2 = \frac{19}{12} \approx 1.5833$

Trapezoidal: $\int_1^2 \left(\frac{x^2}{4} + 1\right) dx \approx \frac{1}{8}\left[\left(\frac{1^2}{4} + 1\right) + 2\left(\frac{(5/4)^2}{4} + 1\right) + 2\left(\frac{(3/2)^2}{4} + 1\right) + 2\left(\frac{(7/4)^2}{4} + 1\right) + \left(\frac{2^2}{4} + 1\right)\right] = \frac{203}{128} \approx 1.5859$

Simpson's: $\int_0^1 \left(\frac{x^2}{4} + 1\right) dx \approx \frac{1}{12}\left[\left(\frac{1^2}{4} + 1\right) + 4\left(\frac{(5/4)^2}{4} + 1\right) + 2\left(\frac{(3/2)^2}{4} + 1\right) + 4\left(\frac{(7/4)^2}{4} + 1\right) + \left(\frac{2^2}{4} + 1\right)\right] = \frac{19}{12} \approx 1.5833$

3. Exact: $\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 4.0000$

Trapezoidal: $\int_0^2 x^3\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^3 + 2(1)^3 + 2\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{17}{4} = 4.2500$

Simpson's: $\int_0^2 x^3\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^3 + 2(1)^3 + 4\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{24}{6} = 4.0000$

4. Exact: $\int_2^3 \frac{2}{x^2}\,dx = \left[-\frac{2}{x}\right]_2^3 = -\frac{2}{3} + \frac{2}{2} = \frac{1}{3}$

Trapezoidal: $\int_2^3 \frac{2}{x^2}\,dx \approx \frac{1}{8}\left[\frac{2}{2^2} + 2\left(\frac{2}{(9/4)^2}\right) + 2\left(\frac{2}{(10/4)^2}\right) + 2\left(\frac{2}{(11/4)^2}\right) + \frac{2}{3^2}\right] \approx 0.3352$

Simpson's: $\int_2^3 \frac{2}{x^2}\,dx \approx \frac{1}{12}\left[\frac{2}{2^2} + 4\left(\frac{2}{(9/4)^2}\right) + 2\left(\frac{2}{(10/4)^2}\right) + 4\left(\frac{2}{(11/4)^2}\right) + \frac{2}{3^2}\right] \approx 0.3334$

5\. Exact: $\int_1^3 x^3\,dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{81}{4} - \frac{1}{4} = 20$

Trapezoidal: $\int_1^3 x^3\,dx \approx \frac{1}{6}\left[1 + 2\left(\frac{4}{3}\right)^3 + 2\left(\frac{5}{3}\right)^3 + 2(2)^3 + 2\left(\frac{7}{3}\right)^3 + 2\left(\frac{8}{3}\right)^3 + 27\right] \approx 20.2222$

Simpson's: $\int_1^3 x^3\,dx \approx \frac{1}{9}\left[1 + 4\left(\frac{4}{3}\right)^3 + 2\left(\frac{5}{3}\right)^3 + 4(2)^3 + 2\left(\frac{7}{3}\right)^3 + 4\left(\frac{8}{3}\right)^3 + 27\right] = 20.0000$

6\. Exact: $\int_0^8 \sqrt[3]{x}\,dx = \left[\frac{3}{4}x^{4/3}\right]_0^8 = 12.0000$

Trapezoidal: $\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{2}\left[0 + 2 + 2\sqrt[3]{2} + 2\sqrt[3]{3} + 2\sqrt[3]{4} + 2\sqrt[3]{5} + 2\sqrt[3]{6} + 2\sqrt[3]{7} + 2\right] \approx 11.7296$

Simpson's: $\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{3}\left[0 + 4 + 2\sqrt[3]{2} + 4\sqrt[3]{3} + 2\sqrt[3]{4} + 4\sqrt[3]{5} + 2\sqrt[3]{6} + 4\sqrt[3]{7} + 2\right] \approx 11.8632$

7\. Exact: $\int_4^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$

Trapezoidal: $\int_4^9 \sqrt{x}\,dx \approx \frac{5}{16}\left[2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3\right] \approx 12.6640$

Simpson's: $\int_4^9 \sqrt{x}\,dx \approx \frac{5}{24}\left[2 + 4\sqrt{\frac{37}{8}} + \sqrt{21} + 4\sqrt{\frac{47}{8}} + \sqrt{26} + 4\sqrt{\frac{57}{8}} + \sqrt{31} + 4\sqrt{\frac{67}{8}} + 3\right] \approx 12.6667$

8\. Exact: $\int_1^4 \left(4 - x^2\right)dx = \left[4x - \frac{x^3}{3}\right]_1^4 = -\frac{16}{3} - \frac{11}{3} = -9$

Trapezoidal: $\int_1^4 \left(4 - x^2\right)dx \approx \frac{1}{4}\left\{3 + 2\left[4 - \left(\frac{3}{2}\right)^2\right] + 2(0) + 2\left[4 - \left(\frac{5}{2}\right)^2\right] + 2(-5) + 2\left[4 - \left(\frac{7}{2}\right)^2\right] - 12\right\} \approx -9.1250$

Simpson's: $\int_1^4 \left(4 - x^2\right)dx \approx \frac{1}{6}\left[3 + 4\left(4 - \frac{9}{4}\right) + 0 + 4\left(4 - \frac{25}{4}\right) - 10 + 4\left(4 - \frac{49}{4}\right) - 12\right] = -9$

9\. Exact: $\int_0^1 \frac{2}{(x+2)^2}\,dx = \left[\frac{-2}{(x+2)}\right]_0^1 = \frac{-2}{3} + \frac{2}{2} = \frac{1}{3}$

Trapezoidal: $\int_0^1 \frac{2}{(x+2)^2}\,dx \approx \frac{1}{8}\left[\frac{1}{2} + 2\left(\frac{2}{((1/4)+2)^2}\right) + 2\left(\frac{2}{((1/2)+2)^2}\right) + 2\left(\frac{2}{((3/4)+2)^2}\right) + \frac{2}{9}\right]$

$$= \frac{1}{8}\left[\frac{1}{2} + 2\left(\frac{32}{81}\right) + 2\left(\frac{8}{25}\right) + 2\left(\frac{32}{121}\right) + \frac{2}{9}\right] \approx 0.3352$$

Simpson's: $\int_0^1 \frac{2}{(x+2)^2}\,dx \approx \frac{1}{12}\left[\frac{1}{2} + 4\left(\frac{2}{((1/4)+2)^2}\right) + 2\left(\frac{2}{((1/2)+2)^2}\right) + 4\left(\frac{2}{((3/4)+2)^2}\right) + \frac{2}{9}\right]$

$$= \frac{1}{12}\left[\frac{1}{2} + 4\left(\frac{32}{81}\right) + 2\left(\frac{8}{25}\right) + 4\left(\frac{32}{121}\right) + \frac{2}{9}\right] \approx 0.3334$$

10. Exact: $\int_0^2 x\sqrt{x^2+1}\,dx = \frac{1}{3}\left[\left(x^2+1\right)^{3/2}\right]_0^2 = \frac{1}{3}\left(5^{3/2}-1\right) \approx 3.393$

Trapezoidal: $\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2+1} + 2(1)\sqrt{1^2+1} + 2\left(\frac{3}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2+1} + 2\sqrt{2^2+1}\right] \approx 3.457$

Simpson's: $\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2+1} + 2(1)\sqrt{1^2+1} + 4\left(\frac{3}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2+1} + 2\sqrt{2^2+1}\right] \approx 3.392$

11. Trapezoidal: $\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{4}\left[1 + 2\sqrt{1+\left(\frac{1}{8}\right)} + 2\sqrt{2} + 2\sqrt{1+\left(\frac{27}{8}\right)} + 3\right] \approx 3.283$

Simpson's: $\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{6}\left[1 + 4\sqrt{1+\left(\frac{1}{8}\right)} + 2\sqrt{2} + 4\sqrt{1+\left(\frac{27}{8}\right)} + 3\right] \approx 3.240$

Graphing utility: 3.241

12. Trapezoidal: $\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{4}\left[1 + 2\left(\frac{1}{\sqrt{1+(1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1+1^3}}\right) + 2\left(\frac{1}{\sqrt{1+(3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.397$

Simpson's: $\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{6}\left[1 + 4\left(\frac{1}{\sqrt{1+(1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1+1^3}}\right) + 4\left(\frac{1}{\sqrt{1+(3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.405$

Graphing utility: 1.402

13. $\int_0^1 \sqrt{x}\sqrt{1-x}\,dx = \int_0^1 \sqrt{x(1-x)}\,dx$

Trapezoidal: $\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{8}\left[0 + 2\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.342$

Simpson's: $\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{12}\left[0 + 4\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 4\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.372$

Graphing utility: 0.393

14. Trapezoidal: $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{16}\left[\sqrt{\frac{\pi}{2}}(1) + 2\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 2\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.430$

Simpson's: $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{24}\left[\sqrt{\frac{\pi}{2}} + 4\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 4\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.458$

Graphing utility: 1.458

15. Trapezoidal: $\int_0^{\sqrt{\pi/2}} \sin\left(x^2\right)dx \approx \frac{\sqrt{\pi/2}}{8}\left[\sin 0 + 2\sin\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\sin\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2\sin\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \sin\left(\sqrt{\frac{\pi}{2}}\right)^2\right] \approx 0.550$

Simpson's: $\int_0^{\sqrt{\pi/2}} \sin\left(x^2\right)dx \approx \frac{\sqrt{\pi/2}}{12}\left[\sin 0 + 4\sin\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\sin\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 4\sin\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \sin\left(\sqrt{\frac{\pi}{2}}\right)^2\right] \approx 0.548$

Graphing utility: 0.549

16. Trapezoidal: $\int_0^{\sqrt{\pi/4}} \tan(x^2)\,dx \approx \frac{\sqrt{\pi/4}}{8}\left[\tan 0 + 2\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 2\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right] \approx 0.271$

Simpson's: $\int_0^{\sqrt{\pi/4}} \tan(x^2)\,dx \approx \frac{\sqrt{\pi/4}}{12}\left[\tan 0 + 4\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 4\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right] \approx 0.257$

Graphing utility: 0.256

17. Trapezoidal: $\int_3^{3.1} \cos x^2\,dx \approx \frac{0.1}{8}\left[\cos(3)^2 + 2\cos(3.025)^2 + 2\cos(3.05)^2 + 2\cos(3.075)^2 + \cos(3.1)^2\right] \approx -0.098$

Simpson's: $\int_3^{3.1} \cos x^2\,dx \approx \frac{0.1}{12}\left[\cos(3)^2 + 4\cos(3.025)^2 + 2\cos(3.05)^2 + 4\cos(3.075)^2 + \cos(3.1)^2\right] \approx -0.098$

Graphing utility: -0.098

18. Trapezoidal: $\int_0^{\pi/2} \sqrt{1+\sin^2 x}\,dx \approx \frac{\pi}{16}\left[1 + 2\sqrt{1+\sin^2\left(\frac{\pi}{8}\right)} + 2\sqrt{1+\sin^2\left(\frac{\pi}{4}\right)} + 2\sqrt{1+\sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{2}\right] \approx 1.910$

Simpson's: $\int_0^{\pi/2} \sqrt{1+\sin^2 x}\,dx \approx \frac{\pi}{24}\left[1 + 4\sqrt{1+\sin^2\left(\frac{\pi}{8}\right)} + 2\sqrt{1+\sin^2\left(\frac{\pi}{4}\right)} + 4\sqrt{1+\sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{2}\right] \approx 1.910$

Graphing utility: 1.910

19. Trapezoidal: $\int_0^{\pi/4} x\tan x\,dx \approx \frac{\pi}{32}\left[0 + 2\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 2\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.194$

Simpson's: $\int_0^{\pi/4} x\tan x\,dx \approx \frac{\pi}{48}\left[0 + 4\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 4\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.186$

Graphing utility: 0.186

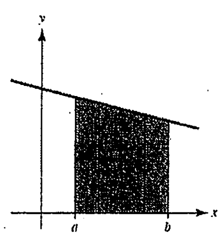
20. Trapezoidal: $\int_0^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{8}\left[1 + \frac{2\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{2\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.836$

Simpson's: $\int_0^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{12}\left[1 + \frac{4\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{4\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.852$

Graphing utility: 1.852

21. Trapezoidal: Linear polynomials
Simpson's: Quadratic polynomials

22. For a linear function, the Trapezoidal Rule is exact. The error formula says that $E \le \frac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$ and $f''(x) = 0$ for a linear function. Geometrically, a linear function is approximated exactly by trapezoids:



23. $f(x) = 2x^3$
$f'(x) = 6x^2$
$f''(x) = 12x$
$f'''(x) = 12$
$f^{(4)}(x) = 0$

(a) Trapezoidal: Error $\le \frac{(3-1)^3}{12(4^2)}(36) = 1.5$ because $\left|f''(x)\right|$ is maximum in $[1, 3]$ when $x = 3$.

(b) Simpson's: Error $\le \frac{(3-1)^5}{180(4^4)}(0) = 0$ because $f^{(4)}(x) = 0$.