4.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding a Sum In Exercises 1–6, find the sum. Use the summation capabilities of a graphing utility to verify your result.

1. $\sum_{i=1}^{6}(3i + 2)$
2. $\sum_{k=3}^{9}(k^2 + 1)$
3. $\sum_{k=0}^{4}\frac{1}{k^2 + 1}$
4. $\sum_{j=4}^{6}\frac{3}{j}$
5. $\sum_{k=1}^{4}c$
6. $\sum_{i=1}^{4}[(i - 1)^2 + (i + 1)^3]$

Using Sigma Notation In Exercises 7–12, use sigma notation to write the sum.

7. $\frac{1}{5(1)} + \frac{1}{5(2)} + \frac{1}{5(3)} + \cdots + \frac{1}{5(11)}$
8. $\frac{9}{1 + 1} + \frac{9}{1 + 2} + \frac{9}{1 + 3} + \cdots + \frac{9}{1 + 14}$
9. $\left[7\left(\frac{1}{6}\right) + 5\right] + \left[7\left(\frac{2}{6}\right) + 5\right] + \cdots + \left[7\left(\frac{6}{6}\right) + 5\right]$
10. $\left[1 - \left(\frac{1}{4}\right)^2\right] + \left[1 - \left(\frac{2}{4}\right)^2\right] + \cdots + \left[1 - \left(\frac{4}{4}\right)^2\right]$
11. $\left[\left(\frac{2}{n}\right)^3 - \frac{2}{n}\right]\left(\frac{2}{n}\right) + \cdots + \left[\left(\frac{2n}{n}\right)^3 - \frac{2n}{n}\right]\left(\frac{2}{n}\right)$
12. $\left[2\left(1 + \frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right) + \cdots + \left[2\left(1 + \frac{3n}{n}\right)^2\right]\left(\frac{3}{n}\right)$

Evaluating a Sum In Exercises 13–20, use the properties of summation and Theorem 4.2 to evaluate the sum. Use the summation capabilities of a graphing utility to verify your result.

13. $\sum_{i=1}^{12}7$
14. $\sum_{i=1}^{30}-18$
15. $\sum_{i=1}^{24}4i$
16. $\sum_{i=1}^{16}(5i - 4)$
17. $\sum_{i=1}^{20}(i - 1)^2$
18. $\sum_{i=1}^{10}(i^2 - 1)$
19. $\sum_{i=1}^{15}i(i - 1)^2$
20. $\sum_{i=1}^{25}(i^3 - 2i)$

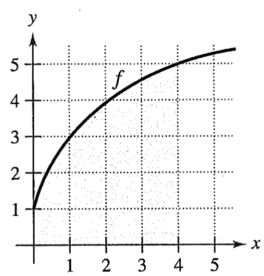
Evaluating a Sum In Exercises 21–24, use the summation formulas to rewrite the expression without the summation notation. Use the result to find the sums for $n = 10, 100, 1000,$ and 10,000.

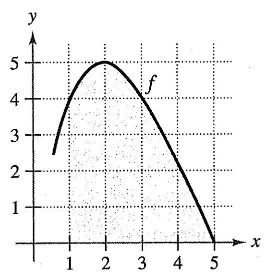
21. $\sum_{i=1}^{n}\frac{2i + 1}{n^2}$
22. $\sum_{j=1}^{n}\frac{7j + 4}{n^2}$
23. $\sum_{k=1}^{n}\frac{6k(k - 1)}{n^3}$
24. $\sum_{i=1}^{n}\frac{2i^3 - 3i}{n^4}$

Approximating the Area of a Plane Region In Exercises 25–30, use left and right endpoints and the given number of rectangles to find two approximations of the area of the region between the graph of the function and the x-axis over the given interval.

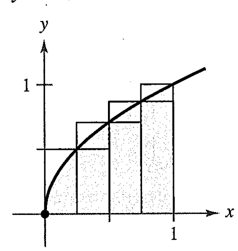
25. $f(x) = 2x + 5$, $[0, 2]$, 4 rectangles
26. $f(x) = 9 - x$, $[2, 4]$, 6 rectangles
27. $g(x) = 2x^2 - x - 1$, $[2, 5]$, 6 rectangles
28. $g(x) = x^2 + 1$, $[1, 3]$, 8 rectangles
29. $f(x) = \cos x$, $\left[0, \frac{\pi}{2}\right]$, 4 rectangles
30. $g(x) = \sin x$, $[0, \pi]$, 6 rectangles

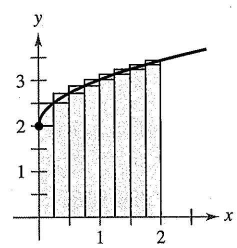
Using Upper and Lower Sums In Exercises 31 and 32, bound the area of the shaded region by approximating the upper and lower sums. Use rectangles of width 1.

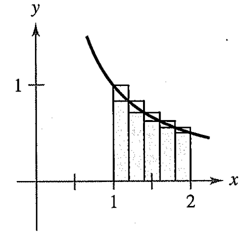
31.


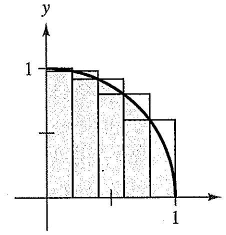
32.


Finding Upper and Lower Sums for a Region In Exercises 33–36, use upper and lower sums to approximate the area of the region using the given number of subintervals (of equal width).

33. $y = \sqrt{x}$


34. $y = \sqrt{x} + 2$


35. $y = \frac{1}{x}$


36. $y = \sqrt{1 - x^2}$


Finding a Limit **In Exercises 37–42, find a formula for the sum of n terms. Use the formula to find the limit as $n \to \infty$.**

37. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{24i}{n^2}$

38. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i}{n}\right)\left(\frac{3}{n}\right)$

39. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(i-1)^2$

40. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$

41. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{i}{n}\right)\left(\frac{2}{n}\right)$

42. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(2+\frac{3i}{n}\right)^3\left(\frac{3}{n}\right)$

43. Numerical Reasoning Consider a triangle of area 2 bounded by the graphs of $y = x$, $y = 0$, and $x = 2$.

(a) Sketch the region.

(b) Divide the interval $[0, 2]$ into n subintervals of equal width and show that the endpoints are

$$0 < 1\left(\frac{2}{n}\right) < \cdots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right).$$

(c) Show that $s(n) = \sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(d) Show that $S(n) = \sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(e) Complete the table.

n	5	10	50	100
$s(n)$				
$S(n)$				

(f) Show that $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n) = 2$.

44. Numerical Reasoning Consider a trapezoid of area 4 bounded by the graphs of $y = x$, $y = 0$, $x = 1$, and $x = 3$.

(a) Sketch the region.

(b) Divide the interval $[1, 3]$ into n subintervals of equal width and show that the endpoints are

$$1 < 1 + 1\left(\frac{2}{n}\right) < \cdots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right).$$

(c) Show that $s(n) = \sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(d) Show that $S(n) = \sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(e) Complete the table.

n	5	10	50	100
$s(n)$				
$S(n)$				

(f) Show that $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n) = 4$.

Finding Area by the Limit Definition **In Exercises 45–54, use the limit process to find the area of the region bounded by the graph of the function and the x-axis over the given interval. Sketch the region.**

45. $y = -4x + 5$, $[0, 1]$

46. $y = 3x - 2$, $[2, 5]$

47. $y = x^2 + 2$, $[0, 1]$

48. $y = 3x^2 + 1$, $[0, 2]$

49. $y = 25 - x^2$, $[1, 4]$

50. $y = 4 - x^2$, $[-2, 2]$

51. $y = 27 - x^3$, $[1, 3]$

52. $y = 2x - x^3$, $[0, 1]$

53. $y = x^2 - x^3$, $[-1, 1]$

54. $y = 2x^3 - x^2$, $[1, 2]$

Finding Area by the Limit Definition **In Exercises 55–60, use the limit process to find the area of the region bounded by the graph of the function and the y-axis over the given y-interval. Sketch the region.**

55. $f(y) = 4y$, $0 \le y \le 2$

56. $g(y) = \frac{1}{2}y$, $2 \le y \le 4$

57. $f(y) = y^2$, $0 \le y \le 5$

58. $f(y) = 4y - y^2$, $1 \le y \le 2$

59. $g(y) = 4y^2 - y^3$, $1 \le y \le 3$

60. $h(y) = y^3 + 1$, $1 \le y \le 2$

Approximating Area with the Midpoint Rule **In Exercises 61–64, use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the graph of the function and the x-axis over the given interval.**

61. $f(x) = x^2 + 3$, $[0, 2]$

62. $f(x) = x^2 + 4x$, $[0, 4]$

63. $f(x) = \tan x$, $\left[0, \frac{\pi}{4}\right]$

64. $f(x) = \cos x$, $\left[0, \frac{\pi}{2}\right]$

WRITING ABOUT CONCEPTS

Approximation **In Exercises 65 and 66, determine which value best approximates the area of the region between the x-axis and the graph of the function over the given interval. (Make your selection on the basis of a sketch of the region, not by performing calculations.)**

65. $f(x) = 4 - x^2$, $[0, 2]$

(a) -2 (b) 6 (c) 10 (d) 3 (e) 8

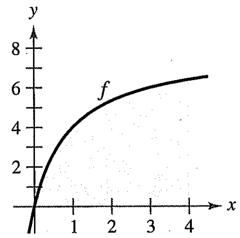
66. $f(x) = \sin \frac{\pi x}{4}$, $[0, 4]$

(a) 3 (b) 1 (c) -2 (d) 8 (e) 6

67. Upper and Lower Sums In your own words and using appropriate figures, describe the methods of upper sums and lower sums in approximating the area of a region.

68. Area of a Region in the Plane Give the definition of the area of a region in the plane.

69. **Graphical Reasoning** Consider the region bounded by the graphs of $f(x) = 8x/(x+1)$, $x = 0$, $x = 4$, and $y = 0$, as shown in the figure. To print an enlarged copy of the graph, go to *MathGraphs.com.*



(a) Redraw the figure, and complete and shade the rectangles representing the lower sum when $n = 4$. Find this lower sum.

(b) Redraw the figure, and complete and shade the rectangles representing the upper sum when $n = 4$. Find this upper sum.

(c) Redraw the figure, and complete and shade the rectangles whose heights are determined by the functional values at the midpoint of each subinterval when $n = 4$. Find this sum using the Midpoint Rule.

(d) Verify the following formulas for approximating the area of the region using n subintervals of equal width.

Lower sum: $s(n) = \sum_{i=1}^{n} f\left[(i-1)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

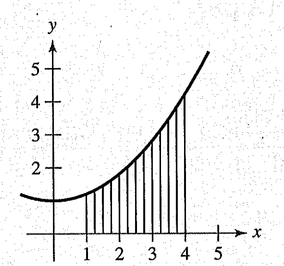
Upper sum: $S(n) = \sum_{i=1}^{n} f\left[(i)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

Midpoint Rule: $M(n) = \sum_{i=1}^{n} f\left[\left(i - \frac{1}{2}\right)\frac{4}{n}\right]\left(\frac{4}{n}\right)$

(e) Use a graphing utility to create a table of values of $s(n)$, $S(n)$, and $M(n)$ for $n = 4, 8, 20, 100$, and 200.

(f) Explain why $s(n)$ increases and $S(n)$ decreases for increasing values of n, as shown in the table in part (e).

70. **HOW DO YOU SEE IT?** The function shown in the graph below is increasing on the interval $[1, 4]$. The interval will be divided into 12 subintervals.



(a) What are the left endpoints of the first and last subintervals?

(b) What are the right endpoints of the first two subintervals?

(c) When using the right endpoints, do the rectangles lie above or below the graph of the function?

(d) What can you conclude about the heights of the rectangles when the function is constant on the given interval?

True or False? **In Exercises 71 and 72, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

71. The sum of the first n positive integers is $n(n+1)/2$.

72. If f is continuous and nonnegative on $[a, b]$, then the limits as $n \to \infty$ of its lower sum $s(n)$ and upper sum $S(n)$ both exist and are equal.

73. **Writing** Use the figure to write a short paragraph explaining why the formula $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ is valid for all positive integers n.

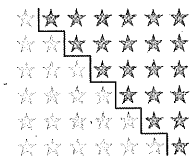


Figure for 73

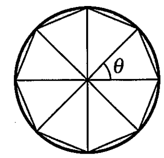


Figure for 74

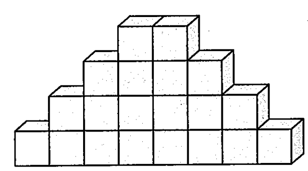
74. **Graphical Reasoning** Consider an n-sided regular polygon inscribed in a circle of radius r. Join the vertices of the polygon to the center of the circle, forming n congruent triangles (see figure).

(a) Determine the central angle θ in terms of n.

(b) Show that the area of each triangle is $\frac{1}{2}r^2 \sin\theta$.

(c) Let A_n be the sum of the areas of the n triangles. Find $\lim_{n\to\infty} A_n$.

75. **Building Blocks** A child places n cubic building blocks in a row to form the base of a triangular design (see figure). Each successive row contains two fewer blocks than the preceding row. Find a formula for the number of blocks used in the design. (*Hint:* The number of building blocks in the design depends on whether n is odd or even.)



n is even.

76. **Proof** Prove each formula by mathematical induction. (You may need to review the method of proof by induction from a precalculus text.)

(a) $\sum_{i=1}^{n} 2i = n(n+1)$ (b) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

PUTNAM EXAM CHALLENGE

77. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Write your answer in the form $(a\sqrt{b} + c)/d$, where a, b, c, and d are integers.

This problem was composed by the Committee on the Putnam Prize Competition.