Ch. 4.2a Notes

I. Sigma Notation

$$\sum_{i=2}^{5} a_i = a_2 + a_3 + a_4 + a_5$$

upper bound

index

lower bound

Ex. 1 $\sum_{i=2}^{4} i^2+1 = (2^2+1)+(3^2+1)+(4^2+1) = \boxed{32}$

II. Summation Formulas *Memorize these*

1) $\sum_{i=1}^{n} 1 = n$

2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2}+\frac{n}{2}$

3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$

4) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$

5) $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

Ex. 2 $\sum_{i=1}^{8}(3i^2+2)$

$= \sum_{i=1}^{8} 3i^2 + \sum_{i=1}^{8} 2$

$= 3\sum_{i=1}^{8} i^2 + 2\sum_{i=1}^{8} 1$

$= 3\cdot\frac{8(9)(17)}{6} + 2\cdot 8$

$= 612 + 16$

$= \boxed{628}$

Ex. 3 $\sum_{i=1}^{10}(i+2)^2$

$= \sum_{i=1}^{10} i^2+4i+4$

$= \sum_{i=1}^{10} i^2 + 4\sum_{i=1}^{10} i + 4\sum_{i=1}^{10} 1$

$= \frac{10(11)(21)}{6} + 4\cdot\frac{10(11)}{2} + 4(10)$

$= 385+220+40$

$= \boxed{645}$

Ex. 4 $\sum_{k=1}^{n}\frac{1}{n}(k^2-1)$

$= \frac{1}{n}\sum_{k=1}^{n} k^2 - \frac{1}{n}\sum_{k=1}^{n} 1$

$= \frac{1}{\cancel{n}}\cdot\frac{\cancel{n}(n+1)(2n+1)}{6} - \frac{1}{\cancel{n}}(\cancel{n})$

$= \frac{(n+1)(2n+1)}{6} - 1$

$= \frac{2n^2+3n+1}{6} - \frac{6}{6}$

$= \boxed{\frac{2n^2+3n-5}{6}}$

4.2a Notes (continued)

III. Limit as n approaches infinity

* Think back about finding horizontal asymptotes.

$\boxed{\text{Ex. 5}}$ If $s(n) = \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$ then find $\lim_{n\to\infty} s(n)$

$$= \lim_{n\to\infty} \frac{n^{2}+\cancel{n}}{2n^{2}} = \boxed{\frac{1}{2}}$$

← Take coefficient if degrees are same in numerator and denominator

$\boxed{\text{Ex. 6}}$ Find $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} 4i$$

$$= \lim_{n\to\infty} \frac{4}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{4}{n^2} \cdot \frac{n(n+1)}{2} = \lim_{n\to\infty} \frac{4n^2+4n}{2n^2}$$

$$= \frac{4}{2} = \boxed{2}$$

$\boxed{\text{Ex. 7}}$ Find $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{4i}{n}+\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n}+\frac{8i}{n^2}+\frac{8i^2}{n^3}\right)$$

$$= \lim_{n\to\infty} \left[\frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \left[\frac{2}{n}(n) + \frac{8}{n^{\cancel{2}}} \cdot \frac{\cancel{n}(n+1)}{2} + \frac{8}{n^{\cancel{3}}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left[\frac{2n}{n} + \frac{8}{n} \cdot \frac{n+1}{2} + \frac{8}{n^2} \cdot \frac{2n^2+3n+1}{6}\right]$$

$$= \lim_{n\to\infty} \left[\frac{2n}{n} + \frac{8n+1}{2n} + \frac{16n^2+24n+8}{6n^2}\right]$$

$$= \frac{2}{1} + \frac{8}{2} + \frac{16}{6}$$

$$= 2 + 4 + 8/3 = \boxed{\frac{26}{3}}$$

Reminder:

* $\boxed{n \text{ is a } \underline{\text{constant}}}$ *

4.2a Selected HW problems

p. 267–268 #1–21 odd, 31, 33, 39, 41, 43

9) Use sigma notation to write the sum:

$$\left[5\left(\tfrac{1}{8}\right)+3\right]+\left[5\left(\tfrac{2}{8}\right)+3\right]+\dots+\left[5\left(\tfrac{8}{8}\right)+3\right]$$

*Notice the incrementing values

$$\boxed{\sum_{i=1}^{8} 5\left(\frac{i}{8}\right)+3}$$

11) $$\left[\left(\frac{2}{n}\right)^3-\frac{2}{n}\right]\left(\frac{2}{n}\right)+\dots\left[\left(\frac{2(2)}{n}\right)^3-\frac{2(2)}{n}\right]\left(\frac{2}{n}\right)+\left[\left(\frac{2n}{n}\right)^3-\frac{2n}{n}\right]\left(\frac{2}{n}\right)$$

$$\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3-\frac{2i}{n}\right]\left(\frac{2}{n}\right) \quad \text{or} \quad \boxed{\frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3-\frac{2i}{n}\right]}$$

17) Use properties of summation to evaluate the sum.

$$\sum_{i=1}^{20}(i-1)^2 = \sum_{i=1}^{20} i^2-2i+1$$

$$= \sum_{i=1}^{20} i^2 - 2\sum_{i=1}^{20} i + \sum_{i=1}^{20} 1$$

$$= \frac{n(n+1)(2n+1)}{6} - 2\cdot\frac{n(n+1)}{2} + n$$

$$= \frac{20(21)(41)}{6} - \cancel{2}\cdot\frac{(20)(21)}{\cancel{2}} + 20$$

$$= 2870 - 420 + 20$$

$$= \boxed{2470}$$

Find $\lim_{n\to\infty} s(n)$

*Compare degrees between numerator and denominator, much like finding horizontal asymptotes

31) $s(n) = \frac{81}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$

$= \lim_{n\to\infty} \frac{81}{n^4}\left[\frac{n^2(n^2+2n+1)}{4}\right]$

$= \lim_{n\to\infty} \frac{81}{n^4}\left[\frac{n^4+2n^3+n^2}{4}\right]$

$= \lim_{n\to\infty} \frac{81n^4+162n^3+81n^2}{4n^4}$

$= \boxed{\frac{81}{4}}$

33) $s(n) = \frac{18}{n^2}\left[\frac{n(n+1)}{2}\right]$

$= \lim_{n\to\infty} \frac{18(n^2+n)}{2n^2}$

$= \lim_{n\to\infty} \frac{18n^2+18n}{2n^2} = \frac{18}{2} = \boxed{9}$

Use summation formulas to find $\lim_{n\to\infty} s(n)$

Formulas:

1) $\sum 1 = n$

2) $\sum i = \frac{n^2}{2}+\frac{n}{2}$

3) $\sum i^2 = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$

4) $\sum i^3 = \frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$

39) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{16i}{n^2}$

*pull coefficients to the front. This will help with the formula substitution step.

*Reminder: n is a constant

$= \lim_{n\to\infty} \frac{16}{n^2} \boxed{\sum_{i=1}^{n} i}$

$= \lim_{n\to\infty} \frac{16}{n^2}\left[\frac{n^2}{2}+\frac{n}{2}\right]$

$= \lim_{n\to\infty} \frac{16n^2}{2n^2} + \cancel{\frac{16n}{2n^2}}$

$= \frac{16}{2} = \boxed{8}$

41) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(i-1)^2$

$= \frac{1}{n^3}\left[\lim_{n\to\infty} \sum_{i=1}^{n} i^2 - 2i + 1\right]$

$= \frac{1}{n^3} \lim_{n\to\infty}\left[\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right]$

$= \frac{1}{n^3} \lim_{n\to\infty}\left[\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} - 2\left[\frac{n^2}{2} + \frac{n}{2}\right] + n\right]$

$= \lim_{n\to\infty}\left[\frac{n^3}{3n^3} + \frac{n^2}{2n^3} + \frac{n}{6n^3} - \frac{2n^2}{2n^3} - \frac{2n}{2n^3} + \frac{n}{n^3}\right]$

$= \boxed{\frac{1}{3}}$

43) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)\left(\frac{2}{n}\right)$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n} + \frac{2i}{n^2}\right)$

$= \lim_{n\to\infty}\left[\frac{2}{n}\sum_{i=1}^{n} 1 + \frac{2}{n^2}\sum_{i=1}^{n} i\right]$

$= \lim_{n\to\infty}\left[\frac{2}{n}(n) + \frac{2}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right)\right]$

$= \lim_{n\to\infty} \frac{2n}{n} + \frac{2n^2}{2n^2} + \cancel{\frac{2n}{2n^2}}$

$= 2 + \frac{2}{2}$

$= 2 + 1$

$= \boxed{3}$