Ch. 4.2a Notes

Key

I. Sigma Notation

upper bound

$$\sum_{i=2}^{5} a_i = a_2 + a_3 + a_4 + a_5$$

index

lower bound

argument expression (formula rule)

***Ex.* 1** $\sum_{i=2}^{4} i^2 + 1 = (2^2+1) + (3^2+1) + (4^2+1) = 5 + 10 + 17 = \boxed{32}$

II. Summation Formulas:

1) $\sum_{i=1}^{n} 1 = n$

2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

5) $\sum_{i=1}^{n} c\boldsymbol{a_i} = c\sum_{i=1}^{n} \boldsymbol{a_i}$

Example 2

$$\sum_{i=1}^{8} (3i^2 + 2) =$$

$$\sum_{i=1}^{8} 3i^2 + \sum_{i=1}^{8} 2$$

$$3\sum_{i=1}^{8} i^2 + 2\sum_{i=1}^{8} 1$$

$$3 \cdot \frac{8(8+1)(16+1)}{6} + 2(8)$$

$$= 612 + 16 = \boxed{628}$$

Example 3

$$\sum_{i=1}^{10} (i+2)^2 =$$

$$\sum (i+2)(i+2)$$

$$\sum i^2 + 4i + 4$$

$$\sum_{i=1}^{10} i^2 + 4\sum_{i=1}^{10} i + 4\sum_{i=1}^{10} 1$$

$$\frac{10(10+1)(20+1)}{6} + 4 \cdot \frac{10(11)}{2} + 4 \cdot 10$$

$$= 385 + 220 + 40 = \boxed{645}$$

Example 4

$$\sum_{k=1}^{n} \frac{1}{n}(k^2 - 1) = \sum \frac{1}{n}k^2 - \frac{1}{n}(1)$$

$$\frac{1}{n}\sum_{k=1}^{n} k^2 - \frac{1}{n}\sum_{k=1}^{n} 1$$

$$\frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{1}{n} \cdot n$$

$$\frac{(n+1)(2n+1)}{6} - 1$$

$$\frac{2n^2+3n+1}{6} - \frac{6}{6} = \boxed{\frac{2n^2+3n-5}{6}}$$

III. Limits as $\boldsymbol{n}$ approaches infinity

*Think back about finding horizontal asymptotes (*compare degrees between numerator and denominator)

Example 5: $If\ S(n) = \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$, $then\ find\ \lim_{n\to\infty} S(n)$

$$\lim_{n\to\infty} \frac{1}{n^2}\cdot\frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} \frac{n(n+1)}{2n^2}$$

$$\lim_{n\to\infty} \frac{n^2+2n}{2n^2} = \boxed{\frac{1}{2}}$$

Example 6: *Find* $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4i}{n^2}$$

$$\lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n} i$$

$$\lim_{n\to\infty}\frac{4}{n^2}\cdot\frac{n(n+1)}{2}$$

$$\lim_{n\to\infty}\frac{4n^2+4n}{2n^2} = \frac{4}{2} = \boxed{2}$$

Example 7: *Find* $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$ $\rightarrow \frac{2}{n}\left(1+\frac{2i}{n}\right)\left(1+\frac{2i}{n}\right) \rightarrow \frac{2}{n}\left(1+\frac{4i}{n}+\frac{4i^2}{n^2}\right)$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2}{n}+\frac{8i}{n^2}+\frac{8i^2}{n^3}\right)$$

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{2}{n}+\sum_{i=1}^{n}\frac{8i}{n^2}+\sum_{i=1}^{n}\frac{8i^2}{n^3}\right)$$

$$\lim_{n\to\infty}\left[\frac{2}{n}\sum_{i=1}^{n}1+\frac{8}{n^2}\sum_{i=1}^{n}i+\frac{8}{n^3}\sum_{i=1}^{n}i^2\right]$$

$$\lim_{n\to\infty}\left[\frac{2}{n}\cdot n+\frac{8}{n^2}\cdot\frac{n(n+1)}{2}+\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty}\frac{2n}{n}+\frac{8n^2+8n}{2n^2}+\frac{16n^3+24n^2+8n}{6n^3}$$

$$= 2+\frac{8}{2}+\frac{16}{6}$$

$$= 2+4+\frac{8}{3} = \boxed{\frac{26}{3}}$$

Formula Sheet:

Summation Formulas:

1) $\sum_{i=1}^{n} 1 = n$

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5) $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

Area using Limit Definition

$$\lim_{n\to\infty}\sum_{i=1}^{n}(width) * f(left\ endpoint + width * i)$$

$$\text{width} = \frac{b-a}{n}$$

Trapezoid Area:

$$Area = \frac{1}{2}w(h_1 + h_2)$$

Integral Formulas:

Power Rule:

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

Trig Integrals:

$$\int \sin u\, du = -\cos u + C \qquad \int \cos u\, du = \sin u + C$$

$$\int \sec^2 u\, du = \tan u + C \qquad \int \sec u \tan u\, du = \sec u + C$$

$$\int \csc^2 u\, du = -\cot u + C \qquad \int \csc u \cot u\, du = -\csc u + C$$