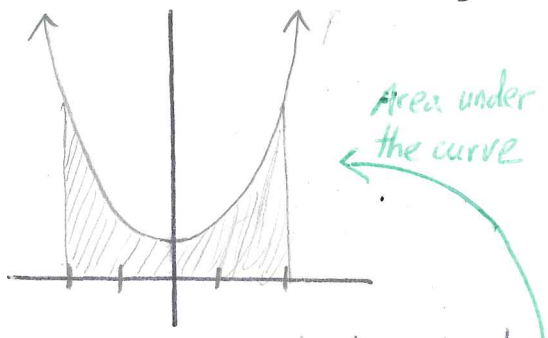


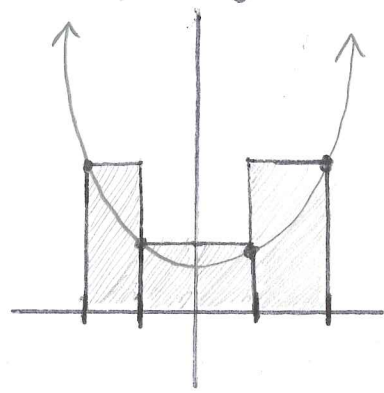
4.2b - Riemann Sums

Riemann Sums - Using rectangles to estimate area of region. (Area under a curve)

Consider the function $f(x) = x^2 + 1$ $[-2, 2]$



1) Upper rectangles or Circumscribed rectangles

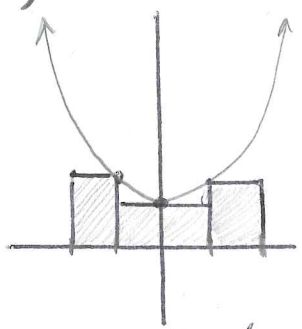


* Using these rectangles will provide an overestimation of the area under the curve.

Suppose we want to estimate the area of the shaded region using a given number of rectangles.

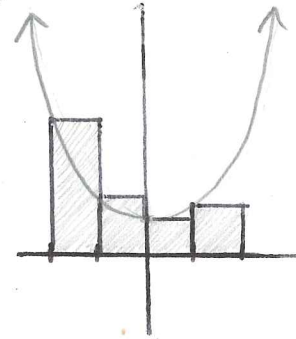
* Notice that one corner of each rectangle is on the graph. This ensures that the height of the rectangle is the same as the value of the function at the point where they connect.

2) Lower or Inscribed rectangles



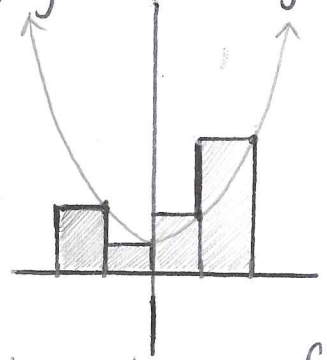
These rectangles provide an underestimation of the area under the curve.

3) Left-handed Rectangles



The left corner of each rectangle attaches to the graph.

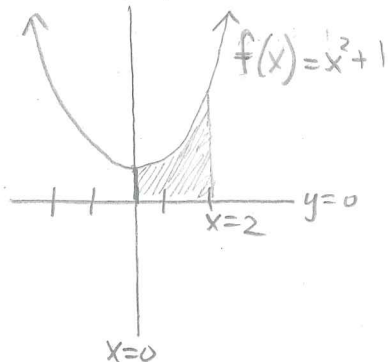
4) Right-handed Rectangles



The right corner of each rectangle attaches to the graph.

4.2b (continued)

Ex. 1 Use 4 rectangles to estimate upper and lower sums for the area bounded by $x=0$, $y=0$, $x=2$, and $y=x^2+1$

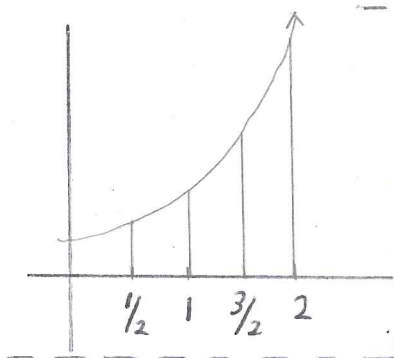


Step 1: Determine width of each rectangle.

$$\text{width} = \frac{b-a}{n}$$

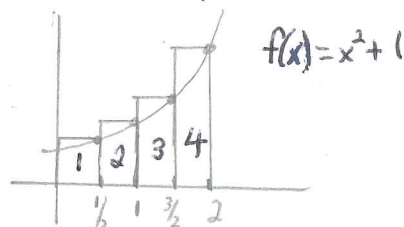
a = left endpoint
 b = right endpoint
 n = number of rectangles

Step 2: Draw the graph. Section off each interval.
Step 3: Find sum of areas of appropriate rectangles.



Since width = $\frac{b-a}{n}$, $b=2$, $a=0$, $n=4$

$$\text{width} = \frac{2-0}{4} = \frac{1}{2}$$



a) Find upper sum

Area = width \times height

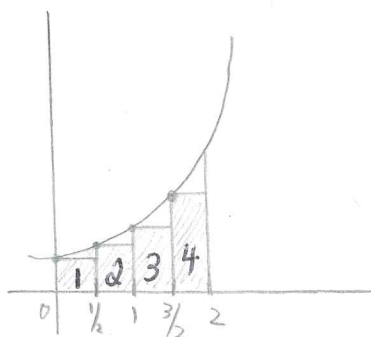
Rectangle #1: Area = $(\frac{1}{2}) \cdot f(\frac{1}{2})$

$$S(4) = \underbrace{\left(\frac{1}{2}\right)}_{\text{Area of Rectangle \#1}} \left(\underbrace{f\left(\frac{1}{2}\right)}_{\text{Area of Rectangle \#2}}\right) + \underbrace{\frac{1}{2}}_{\text{Area of Rectangle \#3}} \underbrace{f(1)}_{\text{Area of Rectangle \#4}} + \underbrace{\frac{1}{2}}_{\text{Area of Rectangle \#3}} \underbrace{f\left(\frac{3}{2}\right)}_{\text{Area of Rectangle \#4}} + \underbrace{\frac{1}{2}}_{\text{Area of Rectangle \#4}} \underbrace{f(2)}_{\text{Area of Rectangle \#4}}$$

$$\begin{aligned}
 f(0.5) &= (0.5)^2 + 1 = 1.25 \\
 f(1) &= 1^2 + 1 = 2 \\
 f(1.5) &= 1.5^2 + 1 = 3.25 \\
 f(2) &= 2^2 + 1 = 5
 \end{aligned}$$

$$S(4) = 0.625 + 1 + 1.625 + 2.5 = \boxed{5.75 \text{ or } \frac{23}{4}}$$

b) Find lower sum



$$\begin{aligned}
 S(4) &= \frac{1}{2}f(0) + \frac{1}{2}f(0.5) + \frac{1}{2}f(1) + \frac{1}{2}f(1.5) & | & f(0) = 0^2 + 1 = 1 \\
 &= 0.5 + 0.625 + 1 + 1.625 \\
 &= \boxed{3.75 \text{ or } \frac{15}{4}}
 \end{aligned}$$

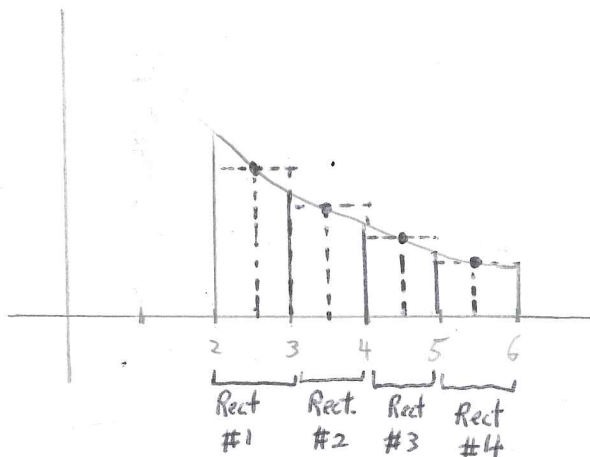
* Good approximation for the actual area is the average between upper and lower sum: $\frac{5.75 + 3.75}{2} = \boxed{4.75}$

4.2b (continued)

Midpoint Rule: Similar to upper/lower sum but use the midpoint of each rectangle to calculate rectangle's height.

Ex. 2 Estimate area under curve $f(x) = \frac{8}{x^2+1}$ from $[2, 6]$
Use midpoint rule with 4 subintervals.

$$\text{width} = \frac{b-a}{n} = \frac{6-2}{4} = \frac{4}{4} = 1$$



$$\text{Midpoint Sum} = \underbrace{1 \cdot f(2.5)}_{\substack{\text{Area} \\ \text{Rect \#1}}} + \underbrace{1 \cdot f(3.5)}_{\substack{\text{Area} \\ \text{Rect \#2}}} + \underbrace{1 \cdot f(4.5)}_{\substack{\text{Area} \\ \text{Rect \#3}}} + \underbrace{1 \cdot f(5.5)}_{\substack{\text{Area} \\ \text{Rect \#4}}}$$

$$f(2.5) = \frac{8}{2.5^2+1} = 1.103$$

$$f(4.5) = \frac{8}{4.5^2+1} = 0.376$$

$$f(3.5) = \frac{8}{3.5^2+1} = 0.604$$

$$f(5.5) = \frac{8}{5.5^2+1} = 0.256$$

$$\text{Midpt Sum} = 1.103 + 0.604 + 0.376 + 0.256 = \boxed{2.34}$$

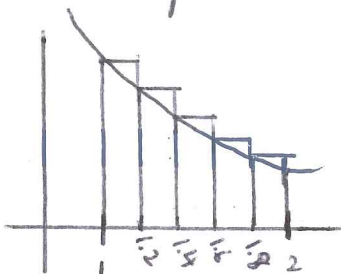
• Why would midpoint sum be a better approximation of area than upper or lower sum?

This is because each rectangle has portions above and below the graph.

* Note: Midpoint sum is not the average between upper and lower sum!

4.26 Selected HW p. 268-269 #23, 25, 27, 29, 63, 65

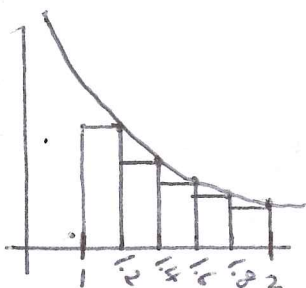
29) $y = \frac{1}{x}$



upper sum:
 $f(1) = 1$
 $f(1.2) = 1/1.2$
 $f(1.4) = 1/1.4$
 $f(1.6) = 1/1.6$
 $f(1.8) = 1/1.8$

upper sum:
 width = $\frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5} = 0.2$

$S = (0.2)f(1) + (0.2)f(1.2) + (0.2)f(1.4) + (0.2)f(1.6) + (0.2)f(1.8)$
 $\approx \boxed{0.746}$

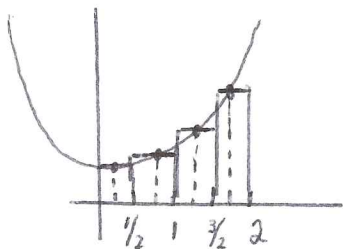


lower sum:

$S = (0.2)f(1.2) + (0.2)f(1.4) + (0.2)f(1.6) + (0.2)f(1.8) + (0.2)f(1)$
 $\approx \boxed{0.646}$

63) Use Midpoint Rule: $n=4$

$f(x) = x^2 + 3$ $[0, 2]$



width = $\frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$

$f(1/4) = \frac{1}{16} + 3$

$f(3/4) = \frac{9}{16} + 3$

$f(5/4) = \frac{25}{16} + 3$

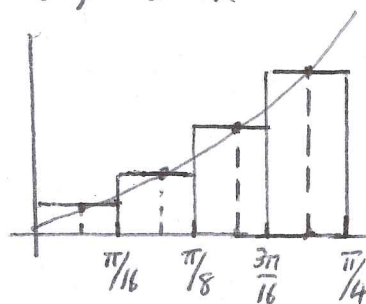
$f(7/4) = \frac{49}{16} + 3$

Area $\approx \frac{1}{2}f(1/4) + \frac{1}{2}f(3/4) + \frac{1}{2}f(5/4) + \frac{1}{2}f(7/4)$
 $\approx \boxed{69/8}$

65) Midpt Rule: $n=4$

$f(x) = \tan x$ $0 \leq x \leq \pi/4$

width = $\frac{\pi/4 - 0}{4} = \frac{\pi}{16}$



$f(\pi/32) = \tan(\pi/32)$

$f(3\pi/32) = \tan(3\pi/32)$

$f(5\pi/32) = \tan(5\pi/32)$

$f(7\pi/32) = \tan(7\pi/32)$

Area $\approx \frac{\pi}{16}f(\pi/32) + \frac{\pi}{16}f(3\pi/32) + \frac{\pi}{16}f(5\pi/32) + \frac{\pi}{16}f(7\pi/32)$

$\approx \boxed{0.345}$