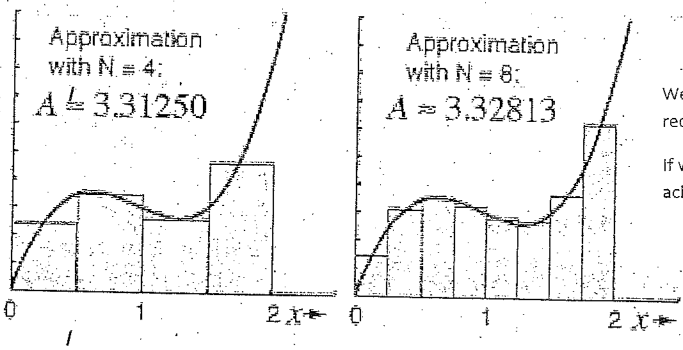
4.2c Finding Exact Area using limits



We can continually improve the Area Approximation under the curve by increasing the number of rectangles : above (n = 4) and n = 8, ...n = 16...

If we let n go out to infinity, (using limits) we will have something better than an approximation, we will achieve the actual area under the curve.

$$\text{Exact Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{b-a}{n}\right) \cdot f\left(a + \left(\frac{b-a}{n}\right)i\right)\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (\text{width}) \cdot f(\underline{\text{left endpoint}} + \underline{\text{width} \cdot i})$$

Memorize → " width f left plus width times i "

$\boxed{\text{Ex. 1}}$ Find exact area between $f(x) = 4 - x^2$ and x-axis from $[-2, 2]$

$$\text{width} = \frac{b-a}{n} = \frac{2--2}{n} = \frac{4}{n}$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \cdot f\left(-2 + \frac{4}{n}i\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \cdot \left[4 - \left(-2 + \frac{4}{n}i\right)^2\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \cdot \left[4 - \left(4 - \frac{8i}{n} - \frac{8i}{n} + \frac{16}{n^2}i^2\right)\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \cdot \left[\cancel{4} - \cancel{4} + \frac{16i}{n} - \frac{16}{n^2}i^2\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \cdot \left[\frac{16i}{n} - \frac{16}{n^2}i^2\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{n^2} \sum_{i=1}^{n} i - \frac{64}{n^3} \sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right) - \frac{64}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{64n^2}{2n^2} + \cancel{\frac{64n}{2n^2}} - \frac{64n^3}{3n^3} - \cancel{\frac{64n^2}{2n^3}} - \cancel{\frac{64n}{6n^3}}\right]$$

$$= \frac{64}{2} - \frac{64}{3}$$

$$= 32 - \frac{64}{3}$$

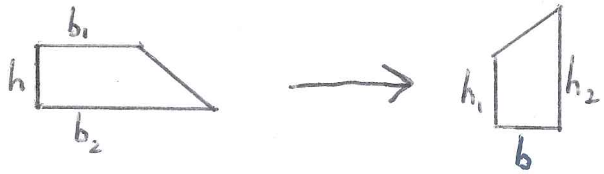
$$= \boxed{\frac{32}{3}}$$

4.6 Trapezoids

* Better approximation than inscribed, circumscribed, or midpoint rectangles.

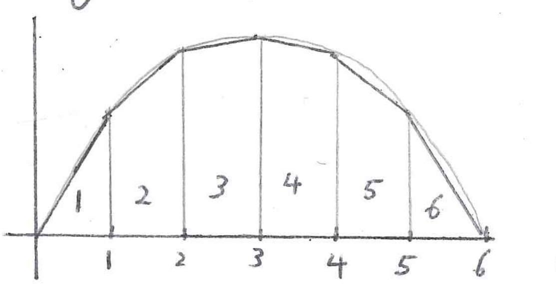
Trapezoidal Rule: Approximate Area of region using areas of trapezoids.

<u>Review</u>: Area of Trapezoid $= \frac{1}{2}h(b_1+b_2)$ or $\frac{h}{2}(b_1+b_2)$



$\boxed{\text{Area} = \frac{b}{2}(h_1+h_2)}$ or $\frac{1}{2}w(h_1+h_2)$

$\boxed{\text{Ex. 1}}$ Estimate area bounded by $f(x)=6x-x^2$ and the x-axis using 6 trapezoids.



Set $6x-x^2=0$ to find bounds for graph

$x(6-x)=0$ $\boxed{x=0, x=6}$

width $= \frac{b-a}{n} = \frac{6-0}{6} = \frac{6}{6} = \boxed{1}$

$\text{Area}_1 = \frac{1}{2}(f(0)+f(1)) = \frac{1}{2}(0+5) = 5/2$

$\text{Area}_2 = \frac{1}{2}(f(1)+f(2)) = \frac{1}{2}(13) = 13/2$

$\text{Area}_3 = \frac{1}{2}(f(2)+f(3)) = \frac{1}{2}(17) = 17/2$

$\text{Area}_4 = \frac{1}{2}(f(3)+f(4)) = \frac{1}{2}(17) = 17/2$

$\text{Area}_5 = \frac{1}{2}(f(4)+f(5)) = \frac{1}{2}(13) = 13/2$

$\text{Area}_6 = \frac{1}{2}(f(5)+f(6)) = \frac{1}{2}(5) = 5/2$

$\text{Area} = \frac{70}{2} = \boxed{35}$

$f(0) = 6(0)-0^2 = 0$

$f(1) = 6(1)-1^2 = 5$

$f(2) = 6(2)-2^2 = 8$

$f(3) = 6(3)-3^2 = 9$

$f(4) = 6(4)-4^2 = 8$

$f(5) = 6(5)-5^2 = 5$

$f(6) = 6(6)-6^2 = 0$

* Integral Notation

$$\int_0^6 (6x-x^2)\,dx$$

6 ← upper bound (b)

$(6x-x^2)dx$ ← function

0 ← lower bound (a)

or

$$A = \frac{1}{2}\left[f(0)+2f(1)+2f(2)+2f(3)+2f(4)+2f(5)+f(6)\right] = \frac{1}{2}(70) = \boxed{35}$$

4.2c/4.6 Selected HW p.269 #47-53 odd

p.314 #1,5,9,13,17

Use limit process to find area of the region

49) $y = x^2+2 \quad [0,1] \quad width = \frac{b-a}{n} = \frac{1-0}{n}$ $\qquad \lim_{n\to\infty} \sum_{i=1}^{n} width \cdot f(left + width \cdot i)$

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f\left(0+\frac{1}{n}i\right)$

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{1}{n}i\right) = \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2+2\right] = \sum_{i=1}^{n} \frac{i^2}{n^3}+\frac{2}{n}$

$\lim_{n\to\infty} \left[\frac{1}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} 1\right] = \lim_{n\to\infty}\left[\frac{1}{n^3}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)+\frac{2}{n}(n)\right]$

$\lim_{n\to\infty} \frac{n^3}{3n^3}+\cancel{\frac{n^2}{2n^3}}+\cancel{\frac{n}{6n^3}}+\frac{2n}{n} = \frac{1}{3}+2 = \boxed{\frac{7}{3}}$

51) $y = 16-x^2 \quad [1,3] \quad width = \frac{3-1}{n} = \frac{2}{n}$

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} f\left(1+\frac{2}{n}i\right) = \sum_{i=1}^{n} \frac{2}{n}\left[16-\left(1+\frac{2}{n}i\right)^2\right] = \sum_{i=1}^{n} \frac{2}{n}\left[16-\left(1+\frac{2}{n}i\right)\left(1+\frac{2}{n}i\right)\right]$

$= \sum_{i=1}^{n} \frac{2}{n}\left[16-\left(1+\frac{4i}{n}+\frac{4}{n^2}i^2\right)\right] = \sum_{i=1}^{n} \frac{2}{n}\left[15-\frac{4i}{n}-\frac{4i^2}{n^2}\right]$

$\lim_{n\to\infty}\left[\frac{30}{n}\sum_{i=1}^{n} 1 - \frac{8}{n^2}\sum_{i=1}^{n} i - \frac{8}{n^3}\sum_{i=1}^{n} i^2\right]$

$\lim_{n\to\infty}\left[\frac{30}{n}(n) - \frac{8}{n^2}\left(\frac{n^2}{2}+\frac{n}{2}\right) - \frac{8}{n^3}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\right]$

$\lim_{n\to\infty}\left[\frac{30n}{n} - \frac{8n^2}{2n^2} - \cancel{\frac{8n}{2n^2}} - \frac{8n^3}{3n^3} - \cancel{\frac{8n^2}{2n^3}} - \cancel{\frac{8n}{6n^3}}\right]$

$30-4-\frac{8}{3} = \boxed{\frac{70}{3}}$

53) $y = 64 - x^3$ $\quad [1, 4] \quad$ width $= \frac{4-1}{n} = \frac{3}{n}$

Pascal's Triangle

1

1 1

1 2 1

$(a+b)^3 \longrightarrow$ 1 3 3 1

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} f\left(1 + \frac{3}{n}i\right)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[64 - \left(1 + \frac{3}{n}i\right)^3\right] = \sum_{i=1} \frac{3}{n}\left[64 - \left(1 + 3\left(\frac{3i}{n}\right) + 3\left(\frac{3}{n}i\right)^2 + \left(\frac{3}{n}i\right)^3\right)\right]$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[63 - \frac{9i}{n} - \frac{27}{n^2}i^2 - \frac{27}{n^3}i^3\right]$$

$$\lim_{n\to\infty}\left[\frac{189}{n}\sum_{i=1}^{n} 1 - \frac{27}{n^2}\sum_{i=1}^{n} i - \frac{81}{n^3}\sum_{i=1}^{n} i^2 - \frac{81}{n^4}\sum_{i=1}^{n} i^3\right]$$

$$\lim_{n\to\infty}\left[\frac{189}{n}(n) - \frac{27}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right) - \frac{81}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) - \frac{81}{n^4}\left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right)\right]$$

exponent degree should never be greater in the numerator.

$$\lim_{n\to\infty}\left[\frac{189n}{n} - \frac{27n^2}{2n^2} - \cancel{\frac{27n}{2n^2}} - \frac{81n^3}{3n^3} - \cancel{\frac{81n^2}{2n^3}} - \cancel{\frac{81n}{6n^3}} - \frac{81n^4}{4n^4} - \cancel{\frac{81n^3}{2n^4}} - \cancel{\frac{81n^2}{4n^4}}\right]$$

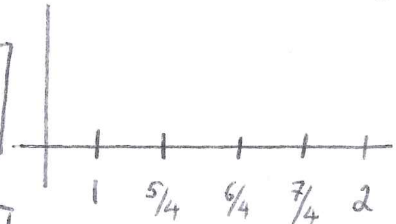
$$= 189 - \frac{27}{2} - \frac{81}{3} - \frac{81}{4} = \boxed{\frac{513}{4} = 128.25}$$

4.6 Trapezoid Rule: p.314 #1, 5, 9, 13, 17

Approximate the definite integral using Trapezoidal Rule with $n=4$

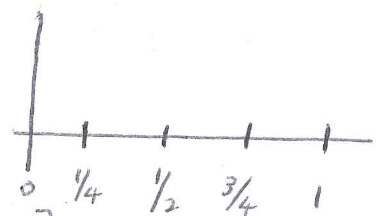
9) $\int_1^2 \frac{1}{(x+1)^2}dx$ width $= \frac{2-1}{4} = \frac{1}{4}$ $\frac{w}{2}\left[h_1 + 2h_2 + 2h_3 + \dots + h_n\right]$

$$A \approx \frac{1}{4}\left(\frac{1}{2}\right)\left[f(1) + 2f(5/4) + 2f(6/4) + 2f(7/4) + f(2)\right]$$



$$= \frac{1}{8}\left[\frac{1}{4} + \frac{32}{81} + \frac{8}{25} + \frac{32}{121} + \frac{1}{9}\right] \approx \boxed{0.1676}$$

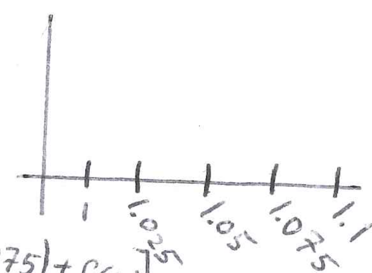
13) $\int_0^1 \sqrt{x}\sqrt{1-x}\,dx$ width $= \frac{1-0}{4} = \frac{1}{4}$



$$A = \frac{1}{4}\left(\frac{1}{2}\right)\left[f(0) + 2f(1/4) + 2f(1/2) + 2f(3/4) + f(1)\right]$$

$$\approx \boxed{0.342}$$

17) $\int_1^{1.1} \sin x^2 dx$ width $= \frac{1.1-1}{4} = \frac{0.1}{4} = \frac{1}{40}$



$$A = \frac{1}{40}\left(\frac{1}{2}\right)\left[f(1) + 2f(1.025) + 2f(1.05) + 2f(1.075) + f(1.1)\right]$$

$$A = \boxed{0.089}$$