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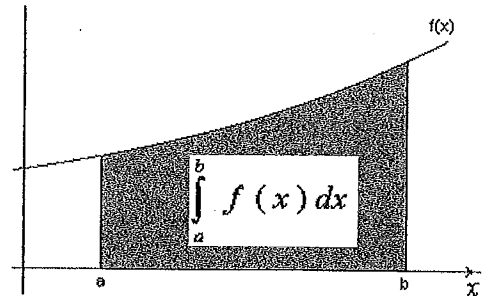
Calculus **Chapters 4.3/4.4a** **1^{st} Fundamental Theorem of Calculus**

Key

$$* \int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is the antiderivative of f.



Recall:

*The general derivative is a **slope-finding function** or formula : (ex. f '(x) = 2x + 1)

*The specific derivative is the **actual slope** at a point (ex: f '(3) = 7)

Likewise…

The indefinite integral is an **Area-Finding Function** or formula (Ex: $\int 2x dx = x^2 + C$)

The definite integral is the **Actual Area** of the region for an interval (Ex: $\int_1^3 2x dx = 8$)

*If a function is **continuous** on a closed interval, then the function is able to be integrated on that interval

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Class Examples:

1. Evaluate $\int_1^4 (3x^2 + 4x - 1)dx$

$$\left.\frac{3x^3}{3} + \frac{4x^2}{2} - x\right]_1^4$$

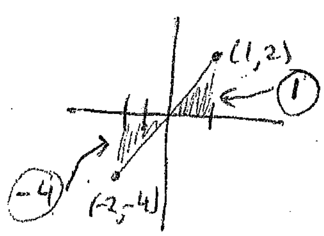
$$\left.x^3 + 2x^2 - x\right]_1^4$$

$$4^3 + 2(4)^2 - 4 - (1^3 + 2 - 1)$$

$$92 - 2 = \boxed{90}$$

For definite integrals:
**NOTE: we don't need to worry about the constant of integration " +C ". It will always wash out.

2. Evaluate $\int_{-2}^1 2x dx = \frac{2x^2}{2} = \left.x^2\right]_{-2}^{1} = 1^2 - (-2)^2 = \boxed{-3}$



portions of graph below x-axis will result in negative value.

Integral Properties:

1) $\int_a^a f(x)dx = 0$

2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ (given that c is between a and b)

Example 3: If $\int_0^3 f(x)dx = 4$ and $\int_3^6 f(x)dx = -1$, find the below:

a) $\int_0^6 f(x)dx = \int_0^3 f(x)dx + \int_3^6 f(x)dx = 4 + (-1) = \boxed{3}$

b) $\int_6^3 f(x)dx = -\int_3^6 f(x)dx = -(-1) = \boxed{1}$

c) $\int_3^3 f(x)dx = \boxed{0}$

d) $\int_3^6 (-5f(x)+3)dx = -5\int_3^6 f(x)dx + \int_3^6 3\,dx$

$3x\Big]_3^6 = 18 - 9 = 9$

$= -5(-1) + 9 = \boxed{14}$

Ex. 4: If $\int_3^8 f'(x)dx = 10$ and $f(8) = 6$, find $f(3)$.

*Reminder that the FFTC can be used as an equation solving tool to find the value of an antiderivative at a specific point. Recall that $\int_a^b f'(x)dx = f(b) - f(a)$

$\int_3^8 f'(x)dx = f(8) - f(3)$

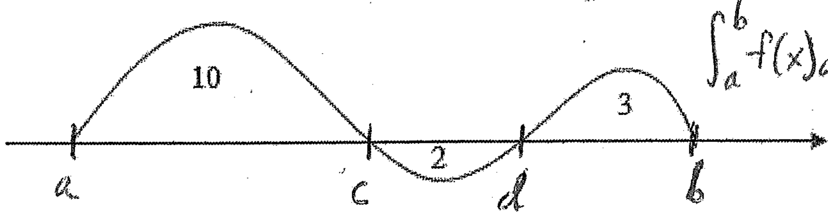
$10 = 6 - f(3)$

$10 - 6 = -f(3)$

$4 = -f(3)$

$\boxed{f(3) = -4}$

Ex. 5: The area for each region is given. Find $\int_a^b f(x)dx$



$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^b f(x)dx$

$= 10 + (-2) + 3$

$= \boxed{11}$

4.3/4.4a (continued)

Integral Properties

1) $\int_a^a f(x)\,dx = 0$

2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

(given that c is between a and b)

$\boxed{\text{Ex. 1}}$ If $\int_0^3 f(x)\,dx = 4$ and $\int_3^6 f(x)\,dx = -1$

find:

a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx$

$= 4 + -1 = \boxed{3}$

b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = -(-1) = \boxed{1}$

c) $\int_3^3 f(x)\,dx = \boxed{0}$

d) $\int_3^6 -5f(x) + 3\,dx$

$= -5\int_3^6 f(x)\,dx + \int_3^6 3\,dx$

$\int_3^6 3\,dx \longrightarrow 3x\Big]_3^6 = 3(6) - 3(3) = 9$

$= -5(-1) + 9 = \boxed{14}$

* Be careful with these types of problems. The integral of 3 is not 3.

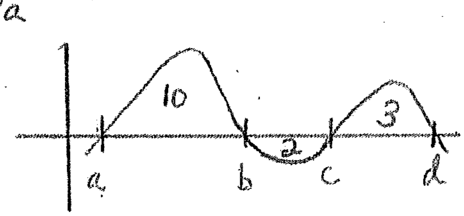
$\boxed{\text{Ex. 2}}$ If $\int_3^8 f'(x)\,dx = 10$ and $f(8) = 6$, find $f(3)$.

Using FFTC, $\int_3^8 f'(x)\,dx = f(8) - f(3)$

$10 = 6 - f(3) \longrightarrow 4 = -f(3)$ so $\boxed{f(3) = -4}$

$\boxed{\text{Ex. 3}}$ The area for each region is shown below.

Find $\int_a^d f(x)\,dx$



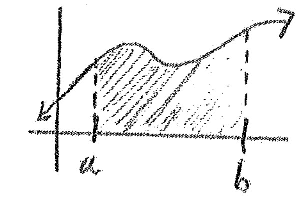
$\int_a^d f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \int_c^d f(x)\,dx$

$= 10 + (-2) + 3$

$= \boxed{11}$

4.3/4.4a 1st Fundamental Theorem of Calulus and PVA

1st Fundamental Theorem of Calculus (FFTC)



$\int_a^b f(x)dx = F(b) - F(a)$ where F is the antiderivative of f.

Recall:

- The general derivative is a slope-finding function (ex. $f'(x) = 2x+1$)
- The specific derivative is the actual slope at a point (ex. $f'(3) = 7$)

Likewise...

- The indefinite integral is an area-finding function
- The definite integral is the actual area of region for an interval

* If a function is continuous on a closed interval, then the function is able to be integrated on that interval.

Ex. 1 Evaluate $\int_1^4 (3x^2+4x-1)dx$

* No need to worry about "+C" It will just naturally cancel out.

$= \frac{3x^3}{3} + \frac{4x^2}{2} - x \Big]_1^4 \rightarrow x^3 + 2x^2 - x \Big]_1^4 = 4^3 + 2(4)^2 - 4 - (1^3 + 2(1)^2 - 1)$

$= 92 - 2 = \boxed{90}$

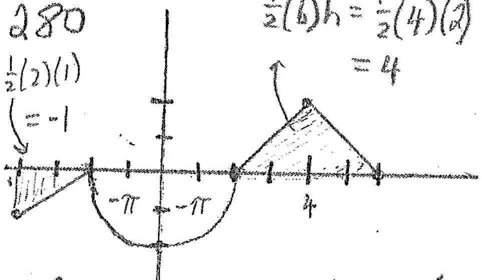
Ex. 2 Evaluate $\int_{-2}^1 2x\,dx$

$= \frac{2x^2}{2}\Big]_{-2}^1 = x^2\Big]_{-2}^1 = 1^2 - ((-2)^2) = 1 - 4 = \boxed{-3}$

* portions of graph below x-axis will result in negative value

4.3-4.4a p.278-280 13-21 odd, 33-39 odd, 47, 49
p. 291-292 5-31 odd

$\boxed{4.3}$ p.280

47) 

$\frac{1}{2}(b)h = \frac{1}{2}(4)(2)$
$=4$

a) $\int_0^2 f(x)dx =$ Area of quarter-circle $A_{circle} = \pi r^2$

$A_{quartercircle} = \frac{1}{4}(\pi r^2) = \frac{1}{4}\pi(2)^2$
$= \frac{4}{4}\pi = \pi$
$= \boxed{-\pi}$

b) $\int_2^6 f(x)dx =$ Area of Triangle $= \boxed{4}$

c) $\int_{-4}^2 f(x)dx = \boxed{-1-2\pi}$

d) $\int_{-4}^6 f(x)dx = -1-2\pi+4 = \boxed{3-2\pi}$

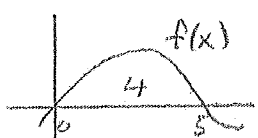
e) $\int_{-4}^6 |f(x)|dx = 1+2\pi+4 = \boxed{5+2\pi}$

f) $\int_{-4}^6 [f(x)+2]dx = \int_{-4}^6 f(x)dx + \int_{-4}^6 2dx$ $\quad 2x\Big]_{-4}^6 = 12-(2(-4)) = 20$

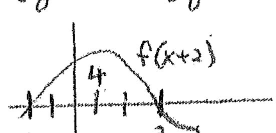
careful with this calculation!

$-1-2\pi+4$
$= 3-2\pi$

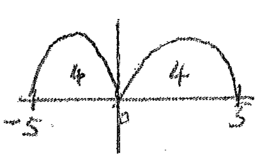
$= 3-2\pi+20$
$= \boxed{23-2\pi}$

49) $\int_0^5 f(x)dx = 4$ 

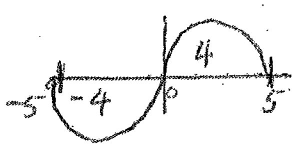
a) $\int_0^5 [f(x)+2] = \int_0^5 f(x)dx + \int_0^5 2dx = 4 + 2x\Big]_0^5 =$ $10-0=10$ $\quad 4+10 = \boxed{14}$

b) $\int_{-2}^3 f(x+2)dx$ 

$= \boxed{4}$

c) $\int_{-5}^5 f(x)dx$ (f is even) 

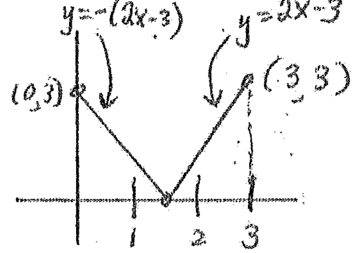
$= \boxed{8}$

d) $\int_{-5}^5 f(x)dx$ (f is odd) 

$= \boxed{0}$

$\boxed{4.4a}$ p. 291-292 5-31 odd

23) $\int_0^3 |2x-3|\,dx$



Geometric method: *Add Areas of triangles $(A=\frac{1}{2}bh)$

$$\frac{1}{2}\left(\frac{3}{2}\right)(3)+\frac{1}{2}\left(\frac{3}{2}\right)(3) = \frac{9}{4}+\frac{9}{4} = \frac{18}{4} = \boxed{\frac{9}{2}}$$

Integral method:

$$\int_0^{1.5} -(2x-3)\,dx + \int_{1.5}^{3} (2x-3)\,dx$$

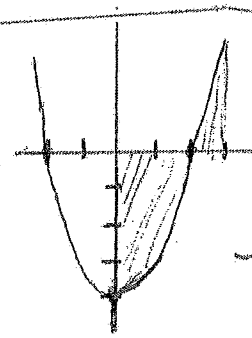
$$\left.-\frac{\cancel{2}x^2}{\cancel{2}}+3x\right]_0^{1.5} = -(1.5)^2+3(1.5) = \underline{2.25}$$

$$\left.\frac{2x^2}{2}-3x\right]_{1.5}^{3} = (9-9)-(1.5^2-3(1.5))$$

$$= -(-2.25) = \underline{2.25}$$

$$2.25+2.25 = 4.5 = \boxed{\frac{9}{2}}$$

25) $\int_0^3 |x^2-4|\,dx$



$$-\left[\int_0^2 x^2-4\,dx\right] + \int_2^3 x^2-4\,dx$$

$$\left.-\frac{x^3}{3}-4x\right]_0^2 \qquad \left.\frac{x^3}{3}-4x\right]_2^3$$

$$-\left(\frac{8}{3}-8\right)-0 = \frac{16}{3} \qquad \frac{27}{3}-12-\left(\frac{8}{3}-8\right) = \frac{7}{3}$$

$$\frac{16}{3}+\frac{7}{3} = \boxed{\frac{23}{3}}$$

27) $\int_0^{\pi} (1+\sin x)\,dx = \left. x-\cos x\right]_0^{\pi} = \pi-\cos\pi-(0-\cos 0)$

$$= \pi-(-1)+1 = \boxed{\pi+2}$$

29) $\int_{-\pi/6}^{\pi/6} \sec^2 x\,dx = \left.\tan x\right]_{-\pi/6}^{\pi/6} = \tan\frac{\pi}{6}-\tan\left(-\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}-\left(-\frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}} = \boxed{\frac{2\sqrt{3}}{3}}$

31) $\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\,d\theta = \left.4\sec\theta\right]_{-\pi/3}^{\pi/3} = 4\sec\left(\frac{\pi}{3}\right)-4\sec\left(-\frac{\pi}{3}\right)$

$$= 4(2)-4(2)$$

$$= 8-8 = \boxed{0}$$

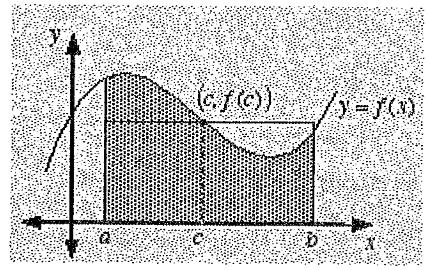
Key

Calculus **Chapter 4.4b** **Average Value Theorem (MVT for Integrals)**

If function f is integrable on the closed interval [a, b], then the average value of f on the interval is

$$f(c) = \frac{1}{b-a}\left[\int_a^b f(x)\,dx\right]$$

height width Area



Since height · width = Area , height = $\frac{\text{Area}}{\text{width}}$

*There exists a rectangle such that the area of the rectangle is the same as the area under the curve (shaded region). f (c) is the height of the rectangle

Example 1: a) Find the average value of $f(x) = x^2 + 1$ on [2, 5]. b) find the c value

* Use Avg. Value Theorem : $f(c) = \frac{1}{5-2}\int_2^5 x^2+1\,dx$

a) $f(c) = \frac{1}{3} \cdot \left.\frac{x^3}{3}+x\right]_2^5 = \frac{1}{3}\left[\frac{5^3}{3}+5-\left(\frac{2^3}{3}+2\right)\right]$

$= \frac{1}{3}\left[\frac{117}{3}+3\right] = \frac{1}{3}\left(\frac{126}{3}\right) = 14$ $\boxed{f(c) = 14}$

b) Find c-value

$f(x) = x^2 + 1$

$f(c) = c^2 + 1$

$14 = c^2 + 1$

$13 = c^2$

$c = \pm\sqrt{13}$

$\boxed{c = \sqrt{13} \text{ since } 2 < \sqrt{13} < 5}$

2nd Fundamental Theorem of Calculus (SFTC)

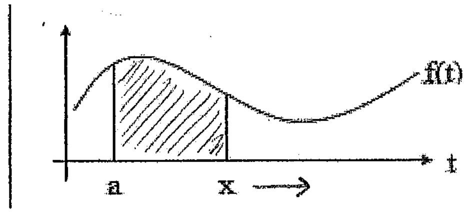
Definite Integral as a Function

To recap, we've covered:

1) Indefinite General Integrals (Area-finding functions)
2) Definite Integrals (Finds Area between 2 x-values)

There is also now a function that is the integral itself. Instead of going from a constant to another constant, we are going from a constant to a moving value of x.

Consider: $f(x) = \int_a^x f(t)\,dt$



<u>2nd Fundamental Theorem of Calculus</u> **Very Important**

Applies the concept that derivative and integrals are inverse operations of each other.

1) $\frac{d}{dx}\left[\int_a^{p(x)} f(t)\,dt\right] = f(p(x))\cdot p'(x)$ (a is a constant)

2) $\frac{d}{dx}\left[\int_{q(x)}^{p(x)} f(t)\,dt\right] = f(p(x))\cdot p'(x) - f(q(x))\cdot q'(x)$

Example 2:

a) $\frac{d}{dx}\left[\int_{-3}^{x} \sqrt{t^2+4}\,dt\right] = \sqrt{x^2+4}\cdot 1$

$= \boxed{\sqrt{x^2+4}}$

b) $\frac{d}{dx}\left[\int_{3}^{x^2} \sqrt{t-1}\,dt\right] = \sqrt{x^2-1}\cdot 2x$

$= \boxed{2x\sqrt{x^2-1}}$

c) $\frac{d}{dx}\left[\int_{10}^{x^2} \sqrt{t-1}\,dt\right] = \sqrt{x^2-1}\cdot 2x$

$= \boxed{2x\sqrt{x^2-1}}$

d) $\frac{d}{dx}\left[\int_{3x}^{0} \frac{1}{t+2}\,dt\right] =$

$\frac{d}{dx}\left[-\int_0^{3x} \frac{1}{t+2}\,dt\right] = -\frac{1}{3x+2}\cdot 3 = \boxed{\frac{-3}{3x+2}}$

e) $\frac{d}{dx}\left[\int_{x}^{x^2} (2t+3)\,dt\right] = \left[2(x^2)+3\right]\cdot(2x) - (2x+3)(1)$

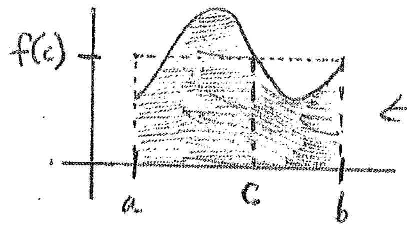
$= 4x^3+6x-2x-3$

$= \boxed{4x^3+4x-3}$

4.4b Average Value Theorem (MVT for Integrals)

If f is able to be integrated on the closed interval $[a,b]$, then the average value of f on the interval is:

$$\underset{\text{height}}{f(c)} = \underset{\text{width}}{\frac{1}{b-a}} \underset{\text{Area}}{\int_a^b f(x)\,dx}$$



c-value exists on the closed interval $[a,b]$

* There exists a rectangle such that the area of the rectangle is the same as the area under the curve. $f(c)$ is the height of rectangle.

Since (height)·(width) = Area, then height $= \frac{\text{Area}}{\text{width}}$

$\boxed{\text{Ex. 1}}$ (a) Find the average value of $f(x) = x^2+1$ on $[2,5]$

Use Avg. Value Theorem: $f(c) = \frac{1}{5-2}\int_2^5 x^2+1\,dx$

$$f(c) = \frac{1}{3}\cdot \frac{x^3}{3}+x\Big]_2^5 = \frac{1}{3}\left[\frac{125}{3}+5-\left(\frac{8}{3}+2\right)\right] = \frac{1}{3}\left[\frac{125}{3}-\frac{8}{3}+5-2\right]$$

$$= \frac{1}{3}\left[\frac{117}{3}+3\right] = \frac{1}{3}\left(\frac{126}{3}\right) = \boxed{14}$$

$\underline{f(c) = 14}$

(b) Find c-value.

$f(x) = x^2+1$

$f(c) = c^2+1$

$14 = c^2+1$

$13 = c^2$

$c = \pm\sqrt{13}$

$\underline{c = \sqrt{13}}$ since $2 < \sqrt{13} < 5$.

4.4b (continued) 2^{nd} Fundamental Theorem of Calculus (SFTC)

Definite Integral as a Function

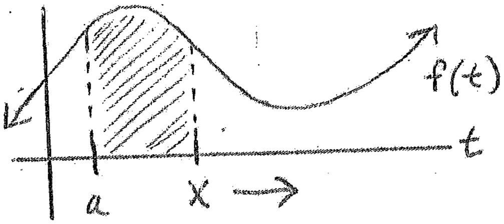
To recap, we've covered

1) (indefinite) general integrals (area-finding functions)
2) definite integrals (finds area between 2 x-values)

Now, there is also a function that is the integral itself.

Consider $f(x) = \int_a^x f(t)\,dt$

Instead of going from a constant to another constant, we are going from a constant to a moving value of x.



2^{nd} Fundamental Theorem of Calculus (SFTC)

$$\frac{d}{dx}\left[\int_a^{p(x)} f(t)\,dt\right] = f(p(x)) \cdot p'(x)$$

Very Important

Ex. 2

a) Find $\frac{d}{dx}\left[\int_{-3}^{x} \sqrt{t^2+4}\,dt\right] = \sqrt{x^2+4} \cdot (1) = \boxed{\sqrt{x^2+4}}$

b) Find $\frac{d}{dx}\left[\int_{3}^{x^2} \sqrt{t-1}\,dt\right] = \sqrt{x^2-1} \cdot (2x) = 2x\sqrt{x^2-1}$

c) Find $\frac{d}{dx}\left[\int_{3x}^{0} \frac{1}{t+2}\,dt\right] = \frac{d}{dx}\left[-\int_0^{3x} \frac{1}{t+2}\,dt\right] = -\frac{1}{3x+2} \cdot 3 = \boxed{\frac{-3}{3x+2}}$

d) Find $\frac{d}{dx}\left[\int_{x}^{x^2} (2t+3)\,dt\right] \rightarrow \frac{d}{dx}\left[\frac{2t^2}{2} + 3t \Big|_{x}^{x^2}\right] =$

SFTC does not apply here. One of the bounds would need to be a constant

$= \frac{d}{dx}\left(\left((x^2)^2 + 3x^2\right) - \left(x^2+3x\right)\right)$

$= \frac{d}{dx}\left(x^4 + 3x^2 - x^2 - 3x\right)$

$= \frac{d}{dx}\left(x^4 + 2x^2 - 3x\right)$

$= \boxed{4x^3 + 4x - 3}$

$\boxed{4.4b}$ Avg. Value Formula p. 291-293
SFTC #33-49 odd, 60, 75-91 odd

Avg. Value Theorem: $\boxed{f(c) = \frac{1}{b-a}\int_a^b f(x)dx}$

43) $f(x) = x - 2\sqrt{x}$ $[0,2]$ $\frac{1}{2-0}\int_0^2 x - 2\sqrt{x}\,dx$

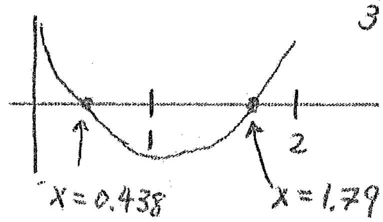
$f(c) = \frac{1}{2}\int_0^2 x - 2x^{1/2}dx = \frac{x^2}{2} - \frac{2x^{3/2}}{3/2} = \frac{x^2}{2} - \frac{2}{3}(2x^{3/2}) = \frac{x^2}{2} - \frac{4}{3}x^{3/2}\Big]_0^2 = \frac{2^2}{2} - \frac{4}{3}(2)^{3/2}$

$= 2 - \frac{4}{3}\sqrt{8}$

$f(c) = \frac{1}{2}\left(2 - \frac{4}{3}\sqrt{8}\right) = 1 - \frac{2}{3}\sqrt{8}$

$c - 2\sqrt{c} = 1 - \frac{2\sqrt{8}}{3} \longrightarrow c - 2\sqrt{c} - 1 + \frac{2\sqrt{8}}{3} = 0$

$\boxed{c = 0.438,\ 1.791}$



45) $f(x) = 2\sec^2 x$ $[-\pi/4, \pi/4]$ $f(c) = \frac{1}{\pi/4 - -\pi/4}\int_{-\pi/4}^{\pi/4} 2\sec^2 x\,dx = \frac{1}{\pi/2}\int_{-\pi/4}^{\pi/4} 2\sec^2 x\,dx$

$f(c) = \frac{2}{\pi}\cdot 2\tan x\Big]_{-\pi/4}^{\pi/4} \rightarrow 2\tan\pi/4 - 2\tan(-\pi/4) = 2(1) - 2(-1) = 2 + 2 = 4$

$f(c) = \frac{2}{\pi}(4) = 8/\pi$ $f(c) = \frac{8}{\pi} \rightarrow 2\sec^2 x = \frac{8}{\pi}$ $\sec^2 x = \frac{4}{\pi}$ $\cos^2 x = \frac{\pi}{4}$

$\cos x = \pm\sqrt{\pi/4}$ $x = \pm\cos^{-1}\left(\sqrt{\pi/4}\right)$, $\boxed{x = \pm 0.4817}$

47) $f(x) = 4 - x^2$ $[-2,2]$

$f(c) = \frac{1}{2--2}\int_{-2}^{2} 4 - x^2 dx$

$f(c) = \frac{1}{4}\int_{-2}^{2} 4 - x^2 dx$

$= \frac{1}{4}\ 4x - \frac{x^3}{3}\Big]_{-2}^{2}$

$= \frac{1}{4}\cdot 8 - 8/3 - (-8 + 8/3)$

$= \frac{1}{4}\cdot 16 - \frac{16}{3}$

$= \frac{1}{4}\left(\frac{32}{3}\right) = \frac{32}{12} \div 4 = \frac{8}{3}$

$f(c) = 8/3$

$4 - c^2 = 8/3$

$-c^2 = 8/3 - 4 = -4/3$

$c^2 = 4/3$

$c = \pm\sqrt{4/3}$

$\boxed{c = \pm 1.155}$

Make sure c-values fall between intervals

4.4b (continued)

49) $f(x) = \sin x \quad [0, \pi]$ $\qquad f(c) = \frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx$

$f(c) = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx$

$= \frac{1}{\pi}\cdot\left(-\cos x\right)\Big]_0^{\pi}$

$= \frac{1}{\pi}\cdot -\cos\pi - (-\cos 0)$

$= \frac{1}{\pi}\cdot[-(-1)+1] = \frac{2}{\pi}$

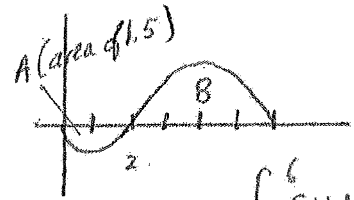
$\sin c = \frac{2}{\pi}$ ← 1st and 2nd quadrant!

$c = 0.69$, $c = \pi - 0.69$

$c = \sin^{-1}\left(\frac{2}{\pi}\right)$

$\boxed{c = 0.69,\ 2.451}$

60) Find avg. value of f over interval [0, 6]



$\int_0^6 f(x)\,dx = 3.5$

$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$ $\qquad f(c) = \frac{1}{6}\cdot 3.5$

$f(c) = \frac{1}{6-0}\int_0^6 f(x)\,dx$ $\qquad = \frac{3.5}{6} = \boxed{\frac{7}{12}}$

#75–91 : SFTC: $\frac{d}{dx}\left[\int_0^x f(t)\,dt\right] = f(x)\cdot x'$

75) $F(x) = \int_0^x (t+2)\,dt$

a) $\int_0^x [t+2]\,dt = \frac{t^2}{2} + 2t\Big]_0^x = \frac{x^2}{2} + 2x$

b) $\frac{d}{dx}\left[\frac{x^2}{2} + 2x\right] = (x+2) \quad = x+2$

83) $F(x) = \int_{-1}^{x} \sqrt{t^4+1}\,dt$ $\quad = (x^4+1)^{1/2}\cdot(1) = \sqrt{x^4+1}$

$\frac{d}{dx}\left[\int_{-1}^{x} (t^4+1)^{1/2}\,dt\right]$

87) $F(x) = \int_x^{x+2} (4t+1)\,dt$

cannot use SFTC shortcut. Neither bound is 0.

$\frac{4t^2}{2} + t\Big]_x^{x+2}$

$= 2(x+2)^2 + x + 2 - (2x^2 + x)$

$2(x^2+4x+4) + x + 2 - 2x^2 - x$

$\cancel{2x^2} + 8x + 8 + \cancel{x} + 2 - \cancel{2x^2} - \cancel{x}$

$= \boxed{8x + 10}$

89) $F(x)=\int_0^{\sin x} \sqrt{t}\,dt$ $\quad = (\sin x)^{1/2} \cdot \cos x$

$\frac{d}{dx}\left[\int_0^{\sin x} t^{1/2}\,dt\right]$ $\quad = \boxed{\cos x \sqrt{\sin x}}$

91) $F(x)=\int_0^{x^3} \sin t^2\,dt$ $\quad = \sin(x^3)^2 \cdot 3x^2$

$\frac{d}{dx}\left[\int_0^{x^2} \sin t^2\,dt\right]$ $\quad = \boxed{3x^2 \sin x^6}$

/

Calculus **Chapter 4.5a** **U-Substitution Method for Indefinite Integrals**

Key

U-Substitution is the reverse of the derivative chain rule. We want to look for a function and its derivative to be in the integral.

*u-substitution is a method of rewriting an integral problem into a simpler one to help us identify an Integral Rule appropriate for the problem.

Example 1:

Suppose $f(x) = \sin(3x)$

$f'(x) = \cos(3x) \cdot 3$
$f'(x) = 3\cos(3x)$

This means that:

$\int 3\cos(3x)dx = \sin(3x) + C$

U-Substitution Steps:

1. Assign the 'u' value to the expression inside the parentheses.
2. Find the derivative of u: $\frac{du}{dx}$
3. Solve for dx.
4. Rewrite the integral in terms of u and du. (Check to make sure no 'x' or 'dx' remains)
5. Evaluate the Integral
6. Write the answer in terms of x.
7. **Not all Integral problems require U-Substitution. Check first to see if expansion/rewriting problem will allow problem to only need the Power Rule**

Ex. 2: $\int x(x^2+1)^{15}dx$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int x \cdot u^{15} \cdot \frac{du}{2x}$

$\int \cancel{x} \cdot u^{15} \cdot \frac{du}{2\cancel{x}}$

$\frac{1}{2}\int u^{15} du = \frac{1}{2} \cdot \frac{u^{16}}{16} + C$

$= \boxed{\frac{1}{32}(x^2+1)^{16} + C}$

Be sure that variable 'x's cancel out. Remaining constants, coefficients are ok.

Ex. 3: $\int x^2 \sec^2(2x^3)dx$

$u = 2x^3$

$\frac{du}{dx} = 6x^2$

$dx = \frac{du}{6x^2}$

$\int \cancel{x^2} \cdot \sec^2 u \cdot \frac{du}{6\cancel{x^2}}$

$\frac{1}{6}\int \sec^2 u \, du$

$= \frac{1}{6}\tan u + C$

$= \boxed{\frac{1}{6}\tan(2x^3) + C}$

Ex. 4: $\int x^3\sqrt{5-x^4}dx = \int x^3(5-x^4)^{1/2}dx$

$u = 5-x^4$

$\frac{du}{dx} = -4x^3$

$dx = \frac{du}{-4x^3}$

$\int \cancel{x^3} \cdot u^{1/2} \cdot \frac{du}{-4\cancel{x^3}}$

$\frac{-1}{4}\int u^{1/2} du$

$= \frac{-1}{4} \frac{u^{3/2}}{3/2} + C$

$= \boxed{-\frac{1}{6}(5-x^4)^{3/2} + C}$

Ex. 5: $\int \tan^5 x \sec^2 x dx$

$\int (\tan x)^5 (\sec x)^2 dx$

$u = \tan x$ | $dx = \frac{du}{\sec^2 x}$

$\frac{du}{dx} = \sec^2 x$

$\int (u)^5 \cdot \cancel{\sec^2 x} \cdot \frac{du}{\cancel{\sec^2 x}} = \int u^5 du$

$= \frac{u^6}{6} + c = \boxed{\frac{1}{6}\tan^6 x + C}$

Ex. 6: $\int (3-y)\left(\frac{1}{\sqrt{y}}\right) dy$

$\int (3-y)(y^{-1/2}) dy$

$\int 3y^{-1/2} - y^{1/2} dy$

$\frac{3y^{1/2}}{1/2} - \frac{y^{3/2}}{3/2} + c$

$\boxed{6y^{1/2} - \frac{2}{3}y^{3/2} + C}$

Change of Variable U-Substitution Method:

Ex. 7: $\int x\sqrt{x+3}dx$

$\int x(x+3)^{1/2} dx$

$u = x+3$

$\frac{du}{dx} = 1$

$dx = du$

*Creative method of substitution in order to eliminate x-variable

$\int x \cdot u^{1/2} du$

$x = u - 3$

$\int (u-3)u^{1/2} du$

$\int u^{3/2} - 3u^{1/2} du$

$\frac{u^{5/2}}{5/2} - \frac{3u^{3/2}}{3/2} + c$

$\boxed{\frac{2}{5}(x+3)^{5/2} - 2(x+3)^{3/2} + C}$

Ex. 8: $\int x^2\sqrt{2-x}dx$

$\int x^2(2-x)^{1/2} dx$

$u = 2-x$

$\frac{du}{dx} = -1$

$dx = -du$

$\int x^2 \cdot u^{1/2} \cdot (-du)$

$x = 2-u$

$-\int (2-u)^2 u^{1/2} du$

$-\int (4-4u+u^2)u^{1/2} du$

$= \int -4u^{1/2} + 4u^{3/2} - u^{5/2}$

$= -\frac{4u^{3/2}}{3/2} + \frac{4u^{5/2}}{5/2} - \frac{u^{7/2}}{7/2} + c$

$\boxed{-\frac{8}{3}(2-x)^{3/2} + \frac{8}{5}(2-x)^{5/2} - \frac{2}{7}(2-x)^{7/2} + C}$

4.5a U-Substitution (Indefinite Integrals)

Ex. 1 Suppose $f(x) = \sin(3x)$

$f'(x) = \cos 3x \cdot 3$

$f'(x) = 3\cos 3x$

This means that:

$$\int 3\cos(3x)\,dx = \sin 3x + C$$

U-Substitution method:

- Reverse of chain rule
- look for function and its derivative in the integral

Steps:

1) Assign "u" to the inside function (value inside parentheses)
2) Find derivative of u : $\frac{du}{dx}$
3) Solve for dx
4) Rewrite integral in terms of u and du (Be sure that no "x" and "dx" remain)
5) Evaluate Integral
6) Present answer in terms of x.

Ex. 2 $\int x(x^2+1)^{15}\,dx$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$2x\,dx = du$

$dx = \frac{du}{2x}$

$\int \cancel{x} \cdot u^{15} \cdot \frac{du}{2\cancel{x}}$

Be sure all "x" cancel out. If not, re-examine problem or use another method.

$= \int u^{15} \cdot \frac{du}{2}$

Include remaining constants in solution (It's ok to have remaining constants)

$= \frac{1}{2}\int u^{15}\,du$

$= \frac{1}{2} \cdot \frac{u^{16}}{16} + C$

$= \frac{1}{32}u^{16} + C$

$= \boxed{\frac{1}{32}(x^2+1)^{16} + C}$

Ex. 3 $\int x^2 \sec^2(2x^3)\,dx$

$u = 2x^3$

$\frac{du}{dx} = 6x^2$

$6x^2\,dx = du$

$dx = \frac{du}{6x^2}$

$\int \cancel{x^2} \cdot \sec^2 u \cdot \frac{du}{6\cancel{x^2}}$

$= \frac{1}{6}\int \sec^2 u\,du$

$= \frac{1}{6}\tan u + C$

$= \boxed{\frac{1}{6}\tan(2x^3) + C}$

4.5a (continued)

Ex. 4 $\int x^3\sqrt{5-x^4}\,dx$

$\int x^3(5-x^4)^{1/2}dx$

$u=5-x^4$

$\frac{du}{dx}=-4x^3$

$-4x^3dx=du$

$dx=\frac{du}{-4x^3}$

$\int \cancel{x^3}\cdot u^{1/2}\cdot\frac{du}{-4\cancel{x^3}}$

$-\frac{1}{4}\int u^{1/2}du$

$-\frac{1}{4}\ \frac{u^{3/2}}{3/2}+C$

$=-\frac{1}{4}\cdot\frac{2}{3}u^{3/2}+C$

$=\boxed{-\frac{1}{6}(5-x^4)^{3/2}+C}$

or $-\frac{1}{6}\sqrt{(5-x^4)^3}+C$

Ex. 5 $\int \tan^5 x\sec^2 x\,dx$

$\int(\tan x)^5(\sec x)^2dx$

$u=\tan x$

$\frac{du}{dx}=\sec^2 x$

$dx=\frac{du}{\sec^2 x}$

$\int u^5\cdot\cancel{\sec^2 x}\cdot\frac{du}{\cancel{\sec^2 x}}$

$\int u^5du$

$=\frac{u^6}{6}+C$

$\frac{1}{6}(\tan x)^6+C$

$=\boxed{\frac{1}{6}\tan^6 x+C}$

Change of Variable U-substitution method

Ex. 6 $\int x\sqrt{x+3}\,dx$

$\int x(x+3)^{1/2}dx$

$u=x+3$

$\frac{du}{dx}=1$

$dx=du$

$\int x\cdot u^{1/2}du$

x does not cancel out.
We can work around this by replacing x in terms of u.

$x=u-3$

$\int(u-3)u^{1/2}du$

$\int u^{3/2}-3u^{1/2}du$

$=\frac{u^{5/2}}{5/2}-3\frac{u^{3/2}}{3/2}+C$

$=\frac{2}{5}u^{5/2}-\cancel{3}\cdot\frac{2}{\cancel{3}}u^{3/2}+C$

$=\boxed{\frac{2}{5}\sqrt{(x+3)^5}-2\sqrt{(x+3)^3}+C}$

$\boxed{4.5a}$ U-substitution p.304-305 #7-33 odd, 43-69 odd

9) $\int \sqrt{9-x^2}\,(-2x)\,dx$ | $u = 9-x^2$, $\frac{du}{dx} = -2x$, $dx = \frac{du}{-2x}$ | $= \int u^{1/2} \cdot \cancel{-2x} \cdot \frac{du}{\cancel{-2x}}$ | $= \frac{u^{3/2}}{3/2} + C$

$= \int (9-x^2)^{1/2}(-2x\,dx)$ | $= \int u^{1/2}\,du$ | $= \frac{2}{3}u^{3/2} + C$

$= \boxed{\frac{2}{3}(9-x^2)^{3/2} + C}$

13) $\int x^2(x^3-1)^4\,dx$ | $\int \cancel{x^2} \cdot u^4 \cdot \frac{du}{3\cancel{x^2}}$ | $= \frac{1}{15}u^5 + C$

$u = x^3 - 1$ | $\frac{1}{3}\int u^4\,du$ | $= \boxed{\frac{1}{15}(x^3-1)^5 + C}$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$ | $\frac{1}{3} \cdot \frac{u^5}{5} + C$

17) $\int 5x\sqrt[3]{1-x^2}\,dx$ | $\int 5\cancel{x} \cdot u^{1/3} \cdot \frac{du}{-2\cancel{x}}$ | $= \frac{-5}{2} \cdot \frac{3}{4}u^{4/3} + C$

$= -15/8\, u^{4/3} + C$

$\int 5x(1-x^2)^{1/3}\,dx$ | $\frac{-5}{2}\int u^{1/3}\,du$ | $= \boxed{\frac{-15}{8}(1-x^2)^{4/3} + C}$

$u = 1-x^2$, $dx = \frac{du}{-2x}$ | $\frac{-5}{2}\left(\frac{u^{4/3}}{4/3}\right) + C$

$\frac{du}{dx} = -2x$

21) $\int \frac{x^2}{(1+x^3)^2}\,dx$ | $= \int \frac{\cancel{x^2}}{u^2} \cdot \frac{du}{3\cancel{x^2}}$ | $\frac{1}{3}\left(\frac{u^{-1}}{-1}\right) + C$

$u = 1 + x^3$ | $= \frac{1}{3}\int \frac{1}{u^2}\,du$ | $\frac{-1}{3u} + C$

$\frac{du}{dx} = 3x^2$ | $= \frac{1}{3}\int u^{-2}\,du$ | $= \boxed{\frac{-1}{3(1+x^3)} + C}$

$dx = \frac{du}{3x^2}$

23) $\int \frac{x}{\sqrt{1-x^2}}\,dx$

$\int \frac{x}{(1-x^2)^{1/2}}\,dx$

$u = 1-x^2$

$\frac{du}{dx} = -2x$

$dx = \frac{du}{-2x}$

$\int \frac{\cancel{x}}{u^{1/2}} \cdot \frac{du}{-2\cancel{x}}$

$\frac{-1}{2}\int \frac{1}{u^{1/2}}\,du$

$= -\frac{1}{2}\int u^{-1/2}\,du$

$= \frac{-1}{2} \cdot \frac{u^{1/2}}{1/2} + C$

$= \frac{-1}{2} \cdot 2u^{1/2} + C$

$= -1u^{1/2} + C$

$= \boxed{-(1-x^2)^{1/2} + C}$ or $-\sqrt{1-x^2} + C$

25) $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt$

$u = 1+\frac{1}{t}$

$u = 1+t^{-1}$

$\frac{du}{dt} = -t^{-2}$

$\frac{du}{dt} = \frac{-1}{t^2}$

$dt = -t^2\,du$

$= \int (u)^3\left(\frac{1}{\cancel{t^2}}\right) \cdot -\cancel{t^2}\,du$

$= -\int u^3\,du$

$= -\frac{u^4}{4} + C$

$= \boxed{\frac{-1}{4}\left(1+\frac{1}{t}\right)^4 + C}$

27) $\int \frac{1}{\sqrt{2x}}\,dx$

$= \int \frac{1}{(2x)^{1/2}}\,dx$

$u = 2x$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$= \int \frac{1}{u^{1/2}} \cdot \frac{du}{2}$

$= \frac{1}{2}\int u^{-1/2}\,du$

$= \frac{1}{2}\left(\frac{u^{1/2}}{1/2}\right) + C$

$= \frac{1}{2} \cdot 2 \cdot u^{1/2} + C$

$= u^{1/2} + C$

$= (2x)^{1/2} + C$

$= \boxed{\sqrt{2x} + C}$

29) $\int \frac{x^2+3x+7}{\sqrt{x}}\,dx$

$= \int \frac{x^2}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{7}{x^{1/2}}\,dx$

$= \int x^{3/2} + 3x^{1/2} + 7x^{-1/2}\,dx$

*No u-substitution needed!

$\frac{x^{5/2}}{5/2} + \frac{3x^{3/2}}{3/2} + \frac{7x^{1/2}}{1/2} + C$

$\frac{2}{5}x^{5/2} + \cancel{3}\left(\frac{2}{\cancel{3}}\right)x^{3/2} + 7(2)x^{1/2} + C$

$= \boxed{\frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C}$

4.5a (continued)

33) $\int (9-y)\sqrt{y}\,dy$

* Expand if possible to avoid u-substitution

$\int 9y^{1/2} - y^{3/2}\,dy$

$= \dfrac{9y^{3/2}}{3/2} - \dfrac{y^{5/2}}{5/2} + C$

$= 9\left(\frac{2}{3}\right)y^{3/2} - \frac{2}{5}y^{5/2} + C$

$= \boxed{6y^{3/2} - \frac{2}{5}y^{5/2} + C}$

47) $\int \frac{1}{\theta^2}\cos\left(\frac{1}{\theta}\right)d\theta$

$u = \frac{1}{\theta} = \theta^{-1}$

$\frac{du}{d\theta} = -1\theta^{-2}$

$\frac{du}{d\theta} = \frac{-1}{\theta^2}$

$d\theta = -\theta^2 du$

$= \int \frac{1}{\cancel{\theta^2}} \cos u \cdot -\cancel{\theta^2}\,du$

$= -\int \cos u\,du$

$= -\sin u + C$

$= \boxed{-\sin\left(\frac{1}{\theta}\right) + C}$

51) $\int \tan^4 x \sec^2 x\,dx$

$= \int (\tan x)^4 \sec^2 x\,dx$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

$= \int u^4 \cancel{\sec^2 x} \cdot \frac{du}{\cancel{\sec^2 x}}$

$= \int u^4\,du$

$= \frac{u^5}{5} + C$

$= \frac{1}{5}(\tan x)^5 + C$

$= \boxed{\frac{1}{5}\tan^5 x + C}$

53) $\int \frac{\csc^2 x}{\cot^3 x}\,dx$

$= \int \frac{\csc^2 x}{(\cot x)^3}\,dx$

$u = \cot x$

$\frac{du}{dx} = -\csc^2 x$

$dx = \frac{du}{-\csc^2 x}$

$= \int \frac{\cancel{\csc^2 x}}{u^3} \cdot \frac{du}{-\cancel{\csc^2 x}}$

$= -\int \frac{1}{u^3}\,du$

$= -\int u^{-3}\,du$

$= -\frac{u^{-2}}{-2} + C$

$= \frac{1}{2u^2} + C$

$= \boxed{\frac{1}{2(\cot x)^2} + C}$

or $\boxed{\frac{1}{2}\tan^2 x + C}$

4.5a (continued)

55) $\int \cot^2 x\,dx$

* u-substitution does not work. Rewrite integrand using trig identities

$(1+\cot^2 x = \csc^2 x)$

$\downarrow$

$\cot^2 x = \csc^2 x - 1$

$= \int \csc^2 x - 1\,dx$

$= \boxed{-\cot x - x + C}$

Find equation for function with given derivative passing through point.

61) $f'(x) = 2x(4x^2-10)^2$ at point $(2,10)$

Steps:
1) Find indefinite integral to get $f(x)$
2) Use ordered pair to solve for C.

$\int 2x(4x^2-10)^2 dx$ $\quad u = 4x^2-10$ $\quad \frac{du}{dx} = 8x$ $\quad dx = \frac{du}{8x}$

$\int \cancel{2x} \cdot u^2 \cdot \frac{du}{\cancel{8x}}$

$\frac{1}{4}\int u^2 du$

$\frac{1}{4}\left(\frac{u^3}{3}\right) + C$

$\frac{1}{12}(4x^2-10)^3 + C$

*plug in $(2,10)$ to find "C"

$f(x) = \frac{1}{12}(4x^2-10)^3 + C$

$10 = \frac{1}{12}(4(2)^2-10)^3 + C$

$10 = \frac{1}{12}(216) + C$

$10 = 18 + C$ $\quad \underline{C = -8}$

$\boxed{f(x) = \frac{1}{12}(4x^2-10)^3 - 8}$

or $\frac{2}{3}(2x^2-5)^3 - 8$

Use change of variable or u-substitution

63) $\int x\sqrt{x+2}\,dx$

$= \int x(x+2)^{1/2} dx$

$u = x+2$

$\frac{du}{dx} = 1$

$dx = du$

$\int x \cdot u^{1/2} du$

$u = x+2$

$x = u-2$

$\int (u-2)u^{1/2} du$

$= \int u^{3/2} - 2u^{1/2} du$

$\frac{u^{5/2}}{5/2} - \frac{2u^{3/2}}{3/2} + C$

$\frac{2}{5}u^{5/2} - 2\left(\frac{2}{3}\right)u^{3/2} + C$

$= \boxed{\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C}$

65) $\int x^2\sqrt{1-x}\,dx$

$= \int x^2(1-x)^{1/2} dx$

$u = 1-x$

$\frac{du}{dx} = -1$

$dx = -du$

$\int x^2 u^{1/2}(-du)$

$u = 1-x$

$x = 1-u$

$\int (1-u)^2 u^{1/2}(-du)$

$-\int (1-2u+u^2)u^{1/2} du$

$\int -u^{1/2} + 2u^{3/2} - u^{5/2} du$

$= \frac{-u^{3/2}}{3/2} + \frac{2u^{5/2}}{5/2} - \frac{u^{7/2}}{7/2} + C$

$= -\frac{2}{3}u^{3/2} + 2\left(\frac{2}{5}\right)u^{5/2} - \frac{2}{7}u^{7/2} + C$

$= \boxed{\frac{-2}{3}(1-x)^{3/2} + \frac{4}{5}(1-x)^{5/2} - \frac{2}{7}(1-x)^{7/2} + C}$

67) $\displaystyle\int \frac{x^2-1}{\sqrt{2x-1}}\,dx$

$\displaystyle\int \frac{x^2-1}{(2x-1)^{1/2}}dx$

$u=2x-1$

$\frac{du}{dx}=2$

$dx=\frac{du}{2}$

$\displaystyle\int \frac{x^2-1}{u^{1/2}}\left(\frac{du}{2}\right)$

$u=2x-1$

$\frac{u+1}{2}=x$

$\displaystyle=\int \frac{\left(\frac{u+1}{2}\right)^2-1}{u^{1/2}}\cdot\frac{du}{2}$

$\displaystyle=\int \frac{\frac{u^2+2u+1}{4}-\frac{4}{4}}{u^{1/2}}\left(\frac{du}{2}\right)$

$\displaystyle=\int \frac{u^2+2u-3}{4u^{1/2}}\left(\frac{du}{2}\right)$

$\displaystyle=\int \frac{u^2+2u-3}{8u^{1/2}}\,du$

$\displaystyle=\int \frac{u^2}{8u^{1/2}}+\frac{2u}{8u^{1/2}}-\frac{3}{8u^{1/2}}\,du$

$\displaystyle=\int \frac{1}{8}u^{3/2}+\frac{1}{4}u^{1/2}-\frac{3}{8}u^{-1/2}\,du$

$=\frac{1}{8}\left(\frac{u^{5/2}}{5/2}\right)+\frac{1}{4}\left(\frac{u^{3/2}}{3/2}\right)-\frac{3}{8}\left(\frac{u^{1/2}}{1/2}\right)$

$=\frac{2}{40}u^{5/2}+\frac{2}{12}u^{3/2}-\frac{6}{8}u^{1/2}+C$

$=\boxed{\frac{1}{20}(2x-1)^{5/2}+\frac{1}{6}(2x-1)^{3/2}-\frac{3}{4}(2x-1)^{1/2}+C}$

69) $\displaystyle\int \frac{-x}{(x+1)-\sqrt{x+1}}\,dx$

$u=x+1$ | $x=u-1$

$\frac{du}{dx}=1$

$dx=du$

$\displaystyle\int \frac{-x}{u-\sqrt{u}}\,du$

$\displaystyle\int \frac{-(u-1)}{u-\sqrt{u}}\,du$

$\displaystyle\int \frac{-u+1}{\sqrt{u}(\sqrt{u}-1)}\,du$

$\displaystyle\int \frac{1-u}{\sqrt{u}(\sqrt{u}-1)}\,du$

$\displaystyle\int \frac{\overset{-1}{\cancel{(1-\sqrt{u})}}(1+\sqrt{u})}{\sqrt{u}\,\cancel{(\sqrt{u}-1)}}\,du$

$\displaystyle\int \frac{-(1+\sqrt{u})}{\sqrt{u}}\,du$

$\displaystyle\int \frac{-1}{\sqrt{u}}-\frac{\sqrt{u}}{\sqrt{u}}\,du$

$\displaystyle=\int -u^{-1/2}-1\,du$

$=\frac{-u^{1/2}}{1/2}-u+C$

$=-2u^{1/2}-u+C$

$=-2(x+1)^{1/2}-(x+1)+C$

$=-2(x+1)^{1/2}-x-1+C$

$=\boxed{-2\sqrt{x+1}-x+C}$

Key

U-Substitution with definite integrals: Be sure the bounds matches the variable of the problem

Ex. 1: $\int_1^2 2x(x^2-2)^3\,dx$

$u = x^2 - 2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int \cancel{2x} \cdot u^3 \cdot \frac{du}{\cancel{2x}}$

$\int u^3\,du$

Convert bounds:

if $x=1$, $u = 1^2 - 2 = -1$

if $x=2$, $u = 2^2 - 2 = 2$

$\int_{-1}^{2} u^3\,du$

$= \frac{u^4}{4}\Big]_{-1}^{2} = \frac{2^4}{4} - \left(\frac{(-1)^4}{4}\right) = \frac{16}{4} - \frac{1}{4}$

$= \boxed{\frac{15}{4}}$

OR:

$\int u^3\,du = \frac{u^4}{4} = \frac{(x^2-2)^4}{4}\Big]_{-1}^{2}$

$= \frac{16}{4} - \frac{1}{4} = \boxed{\frac{15}{4}}$

Ex. 2: $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$

$u = 2x - 1$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$\int \frac{x}{u^{1/2}} \cdot \frac{du}{2}$

* Need to use change of variable method:

$u = 2x - 1$

$\frac{u+1}{2} = x$

$\int \frac{\frac{u+1}{2}}{u^{1/2}} \cdot \frac{du}{2}$

$\frac{1}{4}\int (u+1)u^{-1/2}\,du$

$\frac{1}{4}\int u^{1/2} + u^{-1/2}\,du$

If $x=1$, $u = 2(1)-1 = 1$

if $x=5$, $u = 2(5)-1 = 9$

$\frac{1}{4}\frac{u^{3/2}}{3/2} + \frac{1}{4}\frac{u^{1/2}}{1/2}$

$\frac{1}{6}u^{3/2} + \frac{1}{2}u^{1/2}\Big]_1^9 = \frac{1}{6}(9)^{3/2} + \frac{1}{2}(9)^{1/2} - \left(\frac{1}{6} + \frac{1}{2}\right)$

$= \frac{1}{6}(27) + \frac{1}{2}(3) - \frac{1}{6} - \frac{1}{2}$

$= \boxed{\frac{16}{3}}$

OR $\frac{1}{6}u^{3/2} + \frac{1}{2}u^{1/2}$

$= \frac{1}{6}(2x-1)^{3/2} + \frac{1}{2}(2x-1)^{1/2}\Big]_1^5$

$= \frac{1}{6}(9)^{3/2} + \frac{1}{2}(9)^{1/2} - \left(\frac{1}{6} + \frac{1}{2}\right) = \boxed{\frac{16}{3}}$

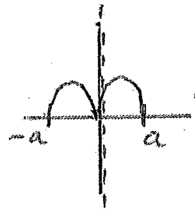
Integrals of Odd and Even Functions

Review: Suppose $\int_{10}^{3} f(x)dx = 9$ and $\int_{-1}^{3} f(x)dx = 5$, find $\int_{-1}^{10} f(x)dx$

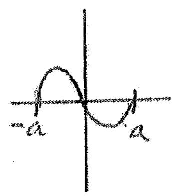
$$\int_{-1}^{10} f(x)dx = \int_{-1}^{3} f(x)dx + \int_{3}^{10} f(x)dx = 5 + (-9) = \boxed{-4}$$

Even/Odd Rules:

Even: $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

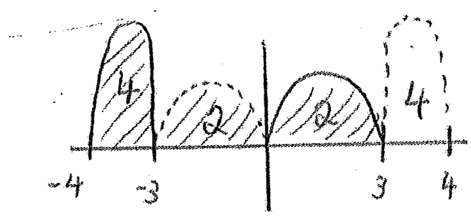


Odd: $\int_{-a}^{a} f(x)dx = 0$



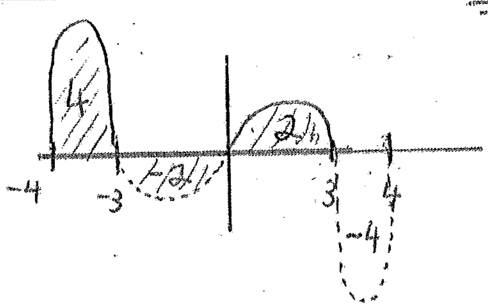
Ex. 3: Suppose g(x) is an even function where $\int_{0}^{3} g(x)dx = 2$ and $\int_{-4}^{-3} g(x)dx = 4$. Find $\int_{-4}^{3} g(x)dx$.

(Sketch a possible graph using the above given information)



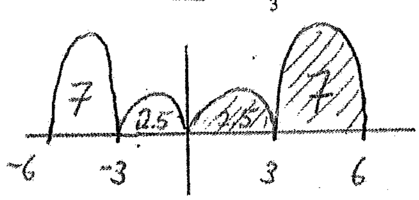
$$\int_{-4}^{3} g(x)dx = 4 + 2 + 2 = \boxed{8}$$

Ex. 4: Same as Example 3, but g(x) is an odd function: $\int_{0}^{3} g(x)dx = 2$ and $\int_{-4}^{-3} g(x)dx = 4$. Find $\int_{-4}^{3} g(x)dx$.



$$\int_{-4}^{3} g(x)dx = \boxed{4}$$

Ex. 5: If f(x) is even and $\int_{3}^{6} f(x)dx = 7$ and $\int_{-6}^{3} f(x)dx = 12$, find $\int_{0}^{6} f(x)dx$



$$\int_{0}^{6} f(x)dx = 2.5 + 7 = \boxed{9.5}$$

4.5b U-Substitution (Definite Integrals)

- Convert bounds to be in terms of u.

Ex. 1 $\displaystyle\int_1^2 2x(x^2-2)^3\,dx$

$u = x^2 - 2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\displaystyle\int \cancel{2x}\cdot u^3 \cdot \frac{du}{\cancel{2x}}$

$\displaystyle\int u^3\,du$

convert bounds:

if $x=1$, $u = 1^2-2=-1$

if $x=2$, $u=2^2-2=2$

$\displaystyle\int_{-1}^{2} u^3\,du$

$$\left.\frac{u^4}{4}\right]_{-1}^{2} = \frac{2^4}{4} - \left(\frac{(-1)^4}{4}\right)$$

$$= \frac{16}{4} - \frac{1}{4} = \boxed{\frac{15}{4}}$$

* Do not change bounds or variables back in terms of x.

Ex. 2 $\displaystyle\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$

$u = 2x-1$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$\displaystyle\int \frac{x}{u^{1/2}}\cdot\frac{du}{2}$

* We need to use change of variable method since x does not cancel out.

$u = 2x-1$

$\frac{u+1}{2} = x$

$$= \int \frac{\frac{u+1}{2}}{u^{1/2}} \cdot \frac{du}{2}$$

$$= \frac{1}{4}\int u^{-1/2}(u+1)\,du$$

$$= \frac{1}{4}\int u^{1/2} + u^{-1/2}\,du$$

convert bounds:

if $x=1$, $u = 2(1)-1=1$

if $x=5$, $u=2(5)-1=9$

$$= \frac{1}{4}\frac{u^{3/2}}{3/2} + \frac{1}{4}\frac{u^{1/2}}{1/2}$$

$$= \left.\frac{1}{6}u^{3/2} + \frac{1}{2}u^{1/2}\right]_1^9$$

$$= \frac{1}{6}(9)^{3/2} + \frac{1}{2}(9)^{1/2} - \left(\frac{1}{6}(1)^{3/2} + \frac{1}{2}(1)^{1/2}\right)$$

$$= \frac{1}{6}(27) + \frac{1}{2}(3) - \frac{1}{6} - \frac{1}{2}$$

$$= \frac{9}{2} + \frac{3}{2} - \frac{1}{6} - \frac{1}{2}$$

$$= \boxed{\frac{16}{3}}$$

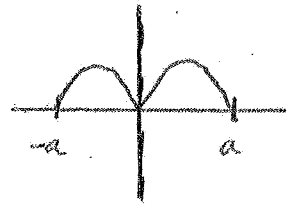
4.5b (continued) Integrals of odd and even functions

Reminder: Suppose $\int_{10}^{3} f(x)\,dx = 9$, $\int_{-1}^{3} f(x)\,dx = 5$, Find $\int_{-1}^{10} f(x)\,dx$

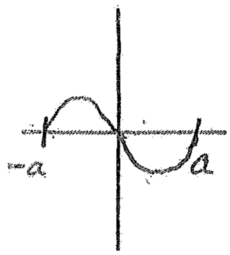
$$\int_{-1}^{10} f(x)\,dx = \int_{-1}^{3} f(x)\,dx + \int_{3}^{10} f(x)\,dx = 5 + (-9) = \boxed{-4}$$

Rules:

Even: $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

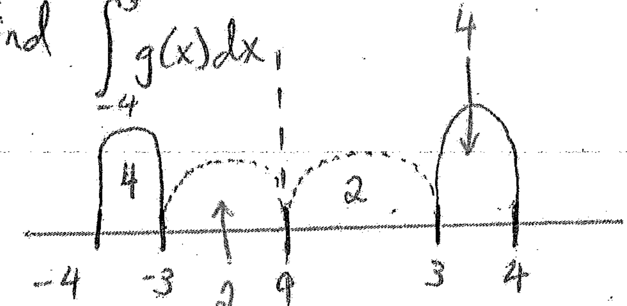


Odd: $\int_{-a}^{a} f(x)\,dx = 0$



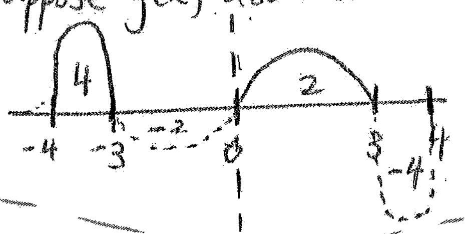
$\boxed{\text{Ex.3}}$ Suppose $g(x)$ is an even function where $\int_{0}^{3} g(x)\,dx = 2$ and $\int_{-4}^{-3} g(x)\,dx = 4$

Find $\int_{-4}^{3} g(x)\,dx$.

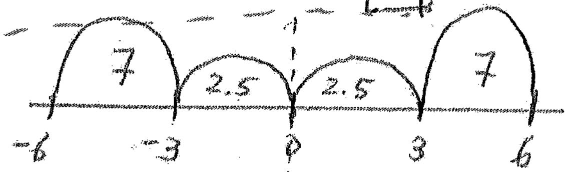


$$\int_{-4}^{3} g(x)\,dx = 4 + 2 + 2 = \boxed{8}$$

$\boxed{\text{Ex.4}}$ Suppose $g(x)$ above is an odd function. Find $\int_{-4}^{3} g(x)\,dx$



$$\int_{-4}^{3} g(x)\,dx = 4 - 2 + 2 = \boxed{4}$$



$\boxed{\text{Ex.5}}$ $f(x)$ is even

$\int_{3}^{6} f(x)\,dx = 7$, $\int_{-6}^{3} f(x)\,dx = 12$. Find $\int_{0}^{6} f(x)\,dx = 2.5 + 7 = \boxed{9.5}$

4.5b U-substitution (definite integrals) p. 305-306 #71-81 odd, 105, 106

73) $\int_1^2 2x^2\sqrt{x^3+1}\,dx$

$= \int_1^2 2x^2(x^3+1)^{1/2}dx$

$u = x^3+1$ | $dx = \frac{du}{3x^2}$

$\frac{du}{dx} = 3x^2$

$\int \cancel{2x^2} \cdot u^{1/2} \cdot \frac{du}{3\cancel{x^2}}$

$\frac{2}{3}\int u^{1/2}du$

$\frac{2}{3}\left(\frac{u^{3/2}}{3/2}\right)$

$\frac{2}{3}\left(\frac{2}{3}\right)u^{3/2} = \frac{4}{9}u^{3/2}$

$u = x^3+1$

for $x=1$, $u = 1+1 = 2$

for $x=2$, $u = 2^3+1 = 9$

$\frac{4}{9}u^{3/2}\Big]_2^9$

$\frac{4}{9}(9)^{3/2} - \frac{4}{9}(2)^{3/2}$

$= \frac{4}{9}(3)^3 - \frac{4}{9}(\sqrt{8})$

$= \frac{4}{\cancel{9}}(\cancel{27}^{3}) - \frac{4}{9}(2\sqrt{2})$

$= \boxed{12 - \frac{8\sqrt{2}}{9}}$

77) $\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}dx$

$u = 1+\sqrt{x}$

$u = 1+x^{1/2}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$dx = 2\sqrt{x}\,du$

$\int \frac{1}{\cancel{\sqrt{x}} \cdot u^2} \cdot 2\cancel{\sqrt{x}}\,du$

$\int \frac{2}{u^2}du$

$\int 2u^{-2}du$

$= \frac{2u^{-1}}{-1} = \frac{-2}{u}$

for $x=1$, $u = 1+\sqrt{1} = 2$

for $x=9$, $u = 1+\sqrt{9} = 4$

$\frac{-2}{u}\Big]_2^4$

$= \frac{-2}{4} - \left(\frac{-2}{2}\right)$

$= \frac{-1}{2} + 1$

$= \boxed{\frac{1}{2}}$

79) $\int_1^2 (x-1)\sqrt{2-x}\,dx$

$\int (x-1)(2-x)^{1/2}dx$

$u = 2-x$

$\frac{du}{dx} = -1$

$dx = -du$

$\int (x-1)u^{1/2}(-du)$

$u = 2-x$

$x = 2-u$

$\int (2-u-1)u^{1/2}(-du)$

$-\int (1-u)u^{1/2}du$

$\int -u^{1/2} + u^{3/2}du$

$-\frac{u^{3/2}}{3/2} + \frac{u^{5/2}}{5/2}$

$-\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}$

$u = 2-x$

for $x=1$, $u = 2-1 = 1$

for $x=2$, $u = 2-2 = 0$

$-\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\Big]_1^0$

$= -\frac{2}{3}(0) + \frac{2}{5}(0) - \left(-\frac{2}{3}(1)^{3/2} + \frac{2}{5}(1)^{5/2}\right)$

$= 0 + \frac{2}{3} - \frac{2}{5}$

$= \frac{10}{15} - \frac{6}{15}$

$= \boxed{\frac{4}{15}}$

$\boxed{4.5b}$ (continued)

81) $\displaystyle\int_0^{\pi/2} \cos\left(\frac{2x}{3}\right)dx$

$u = \frac{2}{3}x \qquad dx = \frac{3du}{2}$

$\frac{du}{dx} = \frac{2}{3}$

$\displaystyle\int \cos u\left(\frac{3}{2}du\right)$

$\displaystyle\frac{3}{2}\int \cos u\,du$

$= \frac{3}{2}\sin u$

$u = \frac{2}{3}x$

for $x=0$, $u = \frac{2}{3}(0) = 0$

for $x = \pi/2$, $u = \frac{2}{3}\left(\pi/2\right) = \pi/3$

$\displaystyle\frac{3}{2}\sin u\Big]_0^{\pi/3}$

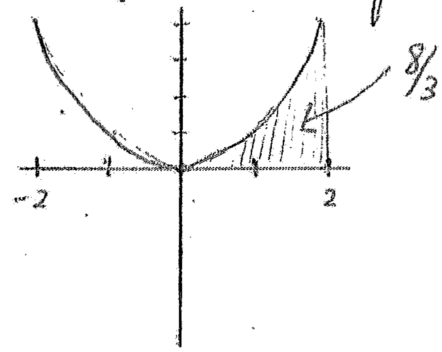
$= \frac{3}{2}\sin(\pi/3) - \frac{3}{2}\sin(0)$

$= \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{3}{2}(0)$

$= \boxed{\frac{3\sqrt{3}}{4}}$

105) Use $\displaystyle\int_0^2 x^2dx = \frac{8}{3}$ to evaluate integral. (Note: $f(x)=x^2$ is an even function)

a) $\displaystyle\int_{-2}^{0} x^2dx = \boxed{\frac{8}{3}}$



b) $\displaystyle\int_{-2}^{2} x^2dx = \frac{8}{3} + \frac{8}{3} = \boxed{\frac{16}{3}}$

c) $\displaystyle\int_0^2 -x^2dx = -\int_0^2 x^2dx = -\left(\frac{8}{3}\right) = \boxed{\frac{-8}{3}}$

d) $\displaystyle\int_{-2}^{0} 3x^2dx = 3\int_{-2}^{0} x^2dx = \cancel{3}\left(\frac{8}{\cancel{3}}\right) = \boxed{8}$

106) Use symmetry of sine and cosine functions to evaluate definite integral

*Recall: $y=\sin x$ is an odd function (origin symmetry)

$y = \cos x$ is an even function (y-axis symmetry)

a) $\displaystyle\int_{-\pi/4}^{\pi/4} \sin x\,dx = \boxed{0}$

b) $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\,dx = 2\left[\int_0^{\pi/4}\cos x\,dx\right] = \sin x\Big]_0^{\pi/4} = \sin(\pi/4) - \sin(0) = \frac{\sqrt{2}}{2}$

$= 2\left(\frac{\sqrt{2}}{2}\right) = \boxed{\sqrt{2}}$

c) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x\,dx = 2\left[\int_0^{\pi/2}\cos x\,dx\right]$

$= \sin x\Big]_0^{\pi/2} = \sin(\pi/2) - \sin(0)$

$= 1$

$= 2\cdot 1 = \boxed{2}$

d) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x\cos x\,dx$

$\downarrow$

$\boxed{0}$

*Test whether $f(x) = \sin x\cos x$ is an even or odd function:

$f(-x) = \sin(-x)\cos(-x)$

$= -\sin x\cdot\cos x = -\sin x\cos x$

Since $f(-x) = -\sin x\cos x$ $(f(-x) = -f(x))$

this is an odd function

(Symmetry about the origin)

Key

Non-AP Calculus Chapter 4.5 Integration and U-Substitution Worksheet

1)

$$\int (5x+4)^5\,dx$$

$u = 5x+4$

$\frac{du}{dx} = 5$

$du = 5dx$

$\frac{du}{5} = dx$

$\int u^5 \cdot \frac{du}{5}$

$\frac{1}{5}\int u^5 du = \frac{1}{5}\left(\frac{u^6}{6}\right) + C$

$= \frac{1}{30}u^6 + C$

$= \boxed{\frac{1}{30}(5x+4)^6 + C}$

2)

$$\int 3t^2(t^3+4)^5\,dt$$

$u = t^3+4$

$\frac{du}{dt} = 3t^2$

$du = 3t^2 dt$

$\frac{du}{3t^2} = dt$

$\int \cancel{3t^2} \cdot u^5 \cdot \frac{du}{\cancel{3t^2}}$

$\int u^5 du = \frac{u^6}{6} + C$

$\boxed{\frac{1}{6}(t^3+4)^6 + C}$

3)

$$\int \sqrt{4x-5}\,dx$$

$\int (4x-5)^{1/2} dx$

$u = 4x-5$

$\frac{du}{dx} = 4$

$du = 4dx$

$\frac{du}{4} = dx$

$\int u^{1/2} \cdot \frac{du}{4}$

$\frac{1}{4}\int u^{1/2} du$

$\frac{1}{4} \cdot \frac{u^{3/2}}{3/2} + C$

$\frac{1}{4} \cdot \frac{2}{3} u^{3/2} + C$

$\frac{2}{12} u^{3/2} + C$

$\boxed{\frac{1}{6}(4x-5)^{3/2} + C}$

4)

$$\int \frac{5x^2}{\sqrt[5]{x^3-2}}dx$$

$\int \frac{5x^2}{(x^3-2)^{1/5}} dx$

$\int 5x^2 (x^3-2)^{-1/5} dx$

$u = x^3-2$

$\frac{du}{dx} = 3x^2$

$du = 3x^2 dx$ $\frac{du}{3x^2} = dx$

$\int 5\cancel{x^2} \cdot u^{-1/5} \cdot \frac{du}{3\cancel{x^2}}$

$\frac{5}{3}\int u^{-1/5} du$

$\frac{5}{3} \cdot \frac{u^{4/5}}{4/5} + C$

$\frac{5}{3} \cdot \frac{5}{4} u^{4/5} + C$

$\boxed{\frac{25}{12}(x^3-2)^{4/5} + C}$

5)

$$\int \cos(2x+1)\,dx$$

$u = 2x+1$

$\frac{du}{dx} = 2$

$du = 2dx$

$\frac{du}{2} = dx$

$\int \cos u \cdot \frac{du}{2}$

$= \frac{1}{2}\int \cos u\, du$

$= \frac{1}{2}\sin u + C$

$\boxed{\frac{1}{2}\sin(2x+1) + C}$

6)

$$\int \sin^{10}(x)\cos(x)\,dx$$

$\int (\sin x)^{10} \cos x\, dx$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\, dx$

$\frac{du}{\cos x} = dx$

$\int u^{10} \cdot \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}}$

$= \int u^{10} du$

$= \frac{u^{11}}{11} + C$

$= \boxed{\frac{1}{11}(\sin x)^{11} + C}$

7. $\int \frac{\sin(x)}{(\cos(x))^5}\,dx$

$\int \frac{\sin x}{(\cos x)^5}\,dx$

$\int \sin x \cdot (\cos x)^{-5}\,dx$

$u = \cos x$ | $du = -\sin x\,dx$

$\frac{du}{dx} = -\sin x$ | $\frac{du}{-\sin x} = dx$

$\int \cancel{\sin x} \cdot u^{-5} \cdot \frac{du}{-\cancel{\sin x}}$

$-\int u^{-5}\,du = \frac{-u^{-4}}{-4} + C$

$\frac{1}{4u^4} + C$

$\boxed{\frac{1}{4(\cos x)^4} + C}$

8)

$\int \frac{2}{\sqrt{3x-7}}\,dx$

$\int \frac{2}{(3x-7)^{1/2}}\,dx$

$\int 2(3x-7)^{-1/2}\,dx$

$u = 3x-7$

$\frac{du}{dx} = 3$

$du = 3dx$ $\quad \frac{du}{3} = dx$

$\int 2 \cdot u^{-1/2} \cdot \frac{du}{3} = \frac{2}{3}\int u^{-1/2}\,du$

$\frac{2}{3} \cdot \frac{u^{1/2}}{1/2} + C = \frac{2}{3} \cdot \frac{2}{1} u^{1/2} + C$

$\boxed{\frac{4}{3}(3x-7)^{1/2} + C}$

9)

$\int \frac{4}{x^2} \sec\left(\frac{5}{x}\right) \tan\left(\frac{5}{x}\right) dx$

$u = \frac{5}{x} = 5x^{-1}$

$\frac{du}{dx} = -5x^{-2}$

$\frac{du}{dx} = \frac{-5}{x^2}$

$x^2 du = -5dx$

$\frac{x^2 du}{-5} = dx$

$\int \frac{4}{\cancel{x^2}} \sec u \tan u \cdot \frac{\cancel{x^2} du}{-5}$

$\frac{-4}{5}\int \sec u \tan u\,du$

$= \frac{-4}{5}\sec u = \boxed{\frac{-4}{5}\sec\left(\frac{5}{x}\right) + C}$

10)

$\int \frac{3x^4}{(7-x^5)^6}\,dx$

$\int 3x^4(7-x^5)^{-6}\,dx$

$u = 7-x^5$

$\frac{du}{dx} = -5x^4$

$du = -5x^4 dx$

$\frac{du}{-5x^4} = dx$

$\int 3\cancel{x^4} \cdot u^{-6} \cdot \frac{du}{-5\cancel{x^4}}$

$\frac{-3}{5}\int u^{-6}\,du$

$= \frac{-3}{5} \cdot \frac{u^{-5}}{-5} + C$

$\frac{3}{25}(7-x^5)^{-5} + C$

$= \boxed{\frac{3}{25(7-x^5)^5} + C}$

11)

$\int \frac{x^3(2x-1)}{\sqrt{x}}\,dx$

$\int (2x^4 - x^3)x^{-1/2}\,dx$

$\int 2x^{7/2} - x^{5/2}\,dx$

$\frac{2x^{9/2}}{9/2} - \frac{x^{7/2}}{7/2} + C$

$2 \cdot \frac{2}{9}x^{9/2} - \frac{2}{7}x^{7/2} + C$

$\boxed{\frac{4}{9}x^{9/2} - \frac{2}{7}x^{7/2} + C}$

12)

$\int 7x^2\sqrt{3-2x^3}\,dx$

$\int 7x^2(3-2x^3)^{1/2}\,dx$

$u = 3-2x^3$

$\frac{du}{dx} = -6x^2$

$du = -6x^2 dx$

$\frac{du}{-6x^2} = dx$

$\int 7\cancel{x^2} \cdot u^{1/2} \cdot \frac{du}{-6\cancel{x^2}}$

$\frac{-7}{6}\int u^{1/2}\,du$

$\frac{-7}{6} \cdot \frac{u^{3/2}}{3/2}$

$\frac{-7}{6} \cdot \frac{2}{3}u^{3/2} + C$

$\frac{-14}{18}u^{3/2} + C$

$= \boxed{\frac{-7}{9}(3-2x^3)^{3/2} + C}$

Practice Problem:

Avg. value theorem: $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

#47) $f(x) = 4 - x^2$ $[-2, 2]$ a) Find avg. value
b) find c-value

$$f(c) = \frac{1}{2-(-2)}\int_{-2}^{2} 4 - x^2\,dx$$

$$f(c) = \frac{1}{4}\int_{-2}^{2} 4 - x^2\,dx$$

$$f(c) = \frac{1}{4} \cdot \left. 4x - \frac{x^3}{3} \right]_{-2}^{2} = 4(2) - \frac{2^3}{3} - \left(4(-2) - \frac{(-2)^3}{3}\right)$$

$$\frac{1}{4}\left[8 - \frac{8}{3} + 8 - \frac{8}{3}\right] = \frac{1}{4}\left[16 - \frac{16}{3}\right] = \frac{1}{4}\left(\frac{32}{3}\right) = \frac{8}{3}$$

a) $\boxed{f(c) = 8/3}$

b) $f(x) = 4 - x^2$

$\frac{8}{3} = 4 - x^2$

$x^2 = 4 - 8/3$

$x^2 = \frac{4}{3}$

$x = \pm\sqrt{\frac{4}{3}}$

$\boxed{c = \frac{2}{\sqrt{3}},\ c = \frac{-2}{\sqrt{3}}}$

in $[-2, 2]$

$\boxed{\text{Ex. 2}}$ Find $\dfrac{d}{dx}\left[\displaystyle\int_{2x^3}^{5} \frac{2t}{5-t^2}\,dt\right]$

$$\overset{*}{\frac{d}{dx}\int_{a}^{p(x)} f(t)\,dt = f\big(p(x)\big)\cdot p'(x)}$$

$$\frac{d}{dx}\int_{5}^{2x^3} \frac{-2t}{5-t^2}\,dt$$

$$= \frac{-2(2x^3)}{5-(2x^3)^2}\cdot 6x^2$$

$$= \boxed{\frac{-24x^5}{5-4x^6}}$$