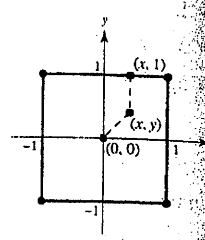
77. Assume that the dartboard has corners at $(\pm 1, \pm 1)$.

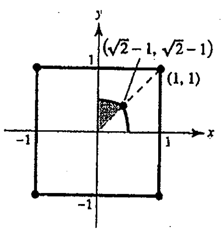
A point (x, y) in the square is closer to the center than the top edge if

$$\sqrt{x^2 + y^2} \le 1 - y$$
$$x^2 + y^2 \le 1 - 2y + y^2$$
$$y \le \tfrac{1}{2}(1 - x^2).$$



By symmetry, a point (x, y) in the square is closer to the center than the right edge if $x \le \tfrac{1}{2}(1 - y^2)$.

In the first quadrant, the parabolas $y = \tfrac{1}{2}(1 - x^2)$ and $x = \tfrac{1}{2}(1 - y^2)$ intersect at $(\sqrt{2} - 1, \sqrt{2} - 1)$. There are 8 equal regions that make up the total region, as indicated in the figure.



Area of shaded region $S = \int_0^{\sqrt{2}-1}\left[\frac{1}{2}(1 - x^2) - x\right]dx = \frac{2\sqrt{2}}{3} - \frac{5}{6}$

Probability $= \frac{8S}{\text{Area square}} = 2\left[\frac{2\sqrt{2}}{3} - \frac{5}{6}\right] = \frac{4\sqrt{2}}{3} - \frac{5}{3}$

Section 4.3 Riemann Sums and Definite Integrals

1. $f(x) = \sqrt{x},\ y = 0,\ x = 0,\ x = 3,\ c_i = \frac{3i^2}{n^2}$

$$\Delta x_i = \frac{3i^2}{n^2} - \frac{3(i-1)^2}{n^2} = \frac{3}{n^2}(2i - 1)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{3i^2}{n^2}}\frac{3}{n^2}(2i - 1)$$
$$= \lim_{n\to\infty}\frac{3\sqrt{3}}{n^3}\sum_{i=1}^{n}(2i^2 - i)$$
$$= \lim_{n\to\infty}\frac{3\sqrt{3}}{n^3}\left[2\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]$$
$$= \lim_{n\to\infty}3\sqrt{3}\left[\frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2}\right]$$
$$= 3\sqrt{3}\left[\frac{2}{3} - 0\right] = 2\sqrt{3} \approx 3.464$$

2. $f(x) = \sqrt[3]{x},\ y = 0,\ x = 0,\ x = 1,\ c_i = \dfrac{i^3}{n^3}$

$$\Delta x_i = \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3} = \frac{3i^2 - 3i + 1}{n^3}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt[3]{\frac{i^3}{n^3}}\left[\frac{3i^2 - 3i + 1}{n^3}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}\left(3i^3 - 3i^2 + i\right)$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[3\left(\frac{n^2(n+1)^2}{4}\right) - 3\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{3n^4 + 6n^3 + 3n^2}{4} - \frac{2n^3 + 3n^2 + n}{2} + \frac{n^2 + n}{2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{3n^4}{4} + \frac{n^3}{2} - \frac{n^2}{4}\right] = \lim_{n\to\infty}\left[\frac{3}{4} + \frac{1}{2n} - \frac{1}{4n^2}\right] = \frac{3}{4}$$

3. $y = 8$ on $[2, 6]$. $\left(\textbf{Note: } \Delta x = \dfrac{6-2}{n} = \dfrac{4}{n},\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(2 + \frac{4i}{n}\right)\left(\frac{4}{n}\right) = \sum_{i=1}^{n} 8\left(\frac{4}{n}\right) = \sum_{i=1}^{n}\frac{32}{n} = \frac{1}{n}\sum_{i=1}^{n} 32 = \frac{1}{n}(32n) = 32$$

$$\int_2^6 8\,dx = \lim_{n\to\infty} 32 = 32$$

4. $y = x$ on $[-2, 3]$. $\left(\textbf{Note: } \Delta x = \dfrac{3-(-2)}{n} = \dfrac{5}{n},\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right)$$

$$= \sum_{i=1}^{n}\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) = -10 + \frac{25}{n^2}\sum_{i=1}^{n} i = -10 + \left(\frac{25}{n^2}\right)\frac{n(n+1)}{2} = -10 + \frac{25}{2}\left(1 + \frac{1}{n}\right) = \frac{5}{2} + \frac{25}{2n}$$

$$\int_{-2}^{3} x\,dx = \lim_{n\to\infty}\left(\frac{5}{2} + \frac{25}{2n}\right) = \frac{5}{2}$$

5. $y = x^3$ on $[-1, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n},\ \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left(-1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right)$$

$$= -2 + \frac{12}{n^2}\sum_{i=1}^{n} i - \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= -2 + 6\left(1 + \frac{1}{n}\right) - 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{2}{n}$$

$$\int_{-1}^{1} x^3\,dx = \lim_{n\to\infty}\frac{2}{n} = 0$$

6. $y = 4x^2$ on $[1, 4]$. $\left(\textbf{Note: } \Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\Delta x_i &= \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \sum_{i=1}^{n} 4\left(1 + \frac{3i}{n}\right)^2\left(\frac{3}{n}\right) \\
&= \frac{12}{n}\sum_{i=1}^{n}\left(1 + \frac{6i}{n} + \frac{9i^2}{n^2}\right) \\
&= \frac{12}{n}\left[n + \frac{6}{n}\frac{n(n+1)}{2} + \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right] \\
&= 12 + 36\frac{n+1}{n} + 18\frac{(n+1)(2n+1)}{n^2} \\
\int_1^4 4x^2\,dx &= \lim_{n\to\infty}\left[12 + \frac{36(n+1)}{n} + \frac{18(n+1)(2n+1)}{n^2}\right] \\
&= 12 + 36 + 36 = 84
\end{aligned}$$

7. $y = x^2 + 1$ on $[1, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\,\Delta x_i &= \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\
&= \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^2 + 1\right]\left(\frac{1}{n}\right) \\
&= \sum_{i=1}^{n}\left[1 + \frac{2i}{n} + \frac{i^2}{n^2} + 1\right]\left(\frac{1}{n}\right) \\
&= 2 + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 2 + \left(1 + \frac{1}{n}\right) + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2} \\
\int_1^2 \left(x^2 + 1\right)dx &= \lim_{n\to\infty}\left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) = \frac{10}{3}
\end{aligned}$$

8. $y = 2x^2 + 3$ on $[-2, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-(-2)}{n} = \dfrac{3}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\Delta x_i &= \sum_{i=1}^{n} f\left(-2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \sum_{i=1}^{n}\left[2\left(-2 + \frac{3i}{n}\right)^2 + 3\right]\left(\frac{3}{n}\right) \\
&= \frac{3}{n}\sum_{i=1}^{n}\left[2\left(4 - \frac{12i}{n} + \frac{9i^2}{n^2}\right) + 3\right] \\
&= \frac{3}{n}\sum_{i=1}^{n}\left[11 - \frac{24i}{n} + \frac{18i^2}{n^2}\right] \\
&= \frac{3}{n}\left[11n - \frac{24}{n}\frac{n(n+1)}{2} + \frac{18}{n^2}\frac{n(n+1)(2n+1)}{6}\right] = 33 - 36\frac{n+1}{n} + 9\frac{(n+1)(2n+1)}{n^2}
\end{aligned}$$

$$\int_{-2}^{1}\left(2x^2 + 3\right)dx = \lim_{n\to\infty}\left[33 - 36\frac{n+1}{n} + 9\frac{(n+1)(2n+1)}{n^2}\right] = 33 - 36 + 18 = 15$$

9. $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} (3c_i + 10)\,\Delta x_i = \int_{-1}^{5} (3x + 10)\,dx$

on the interval $[-1, 5]$.

10. $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} 6c_i(4 - c_i)^2\,\Delta x_i = \int_0^4 6x(4 - x)^2\,dx$

on the interval $[0, 4]$.

11. $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{c_i^2 + 4}\,\Delta x_i = \int_0^3 \sqrt{x^2 + 4}\,dx$

on the interval $[0, 3]$.

12. $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \left(\frac{3}{c_i^2}\right)\Delta x_i = \int_1^3 \frac{3}{x^2}\,dx$

on the interval $[1, 3]$.

13. $\int_0^4 5\,dx$

14. $\int_0^2 (6 - 3x)\,dx$

15. $\int_{-4}^{4} (4 - |x|)\,dx$

16. $\int_0^2 x^2\,dx$

17. $\int_{-5}^{5} (25 - x^2)\,dx$

18. $\int_{-1}^{1} \frac{4}{x^2 + 2}\,dx$

19. $\int_0^{\pi/2} \cos x\,dx$

20. $\int_0^{\pi/4} \tan x\,dx$

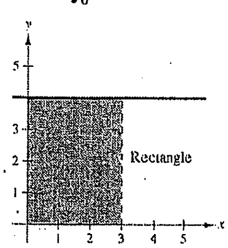
21. $\int_0^2 y^3\,dy$

22. $\int_0^2 (y - 2)^2\,dy$

23. Rectangle

$A = bh = 3(4)$

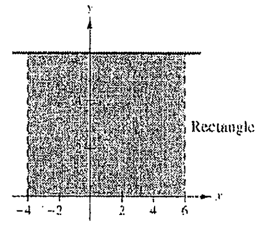
$A = \int_0^3 4\,dx = 12$



24. Rectangle

$A = bh = 10(6) = 60$

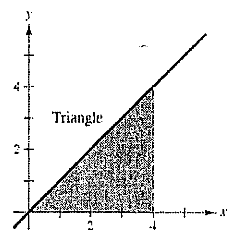
$A = \int_{-4}^{6} 6\,dx = 60$



25. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(4)(4) = 8$

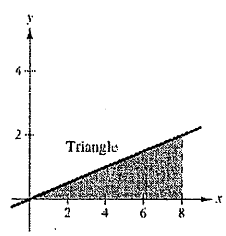
$A = \int_0^4 x\,dx = 8$



26. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(8)(2) = 8$

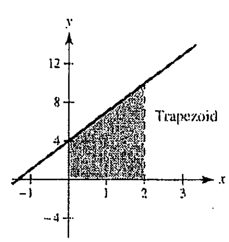
$A = \int_0^8 \frac{x}{4}\,dx = 8$



27. Trapezoid

$A = \frac{b_1 + b_2}{2}h = \left(\frac{4 + 10}{2}\right)2 = 14$

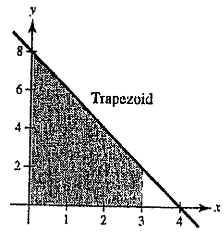
$A = \int_0^2 (3x + 4)\,dx = 14$



28. Trapezoid

$$A = \frac{b_1 + b_2}{2}h = \frac{8 + 2}{2}(3) = 15$$

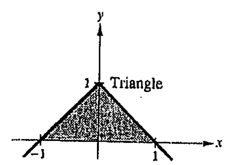
$$A = \int_0^3 (8 - 2x)\,dx = 15$$



29. Triangle

$$A = \tfrac{1}{2}bh = \tfrac{1}{2}(2)(1) = 1$$

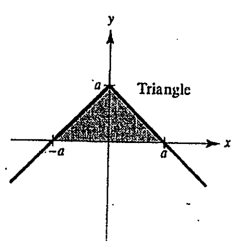
$$A = \int_{-1}^{1} (1 - |x|)\,dx = 1$$



30. Triangle

$$A = \tfrac{1}{2}bh = \tfrac{1}{2}(2a)\,a = a^2$$

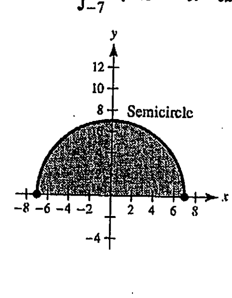
$$A = \int_{-a}^{a} (a - |x|)\,dx = a^2$$



31. Semicircle

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(7)^2 = \frac{49\pi}{2}$$

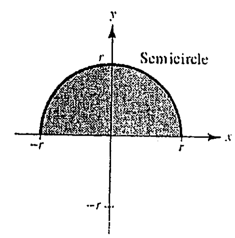
$$A = \int_{-7}^{7} \sqrt{49 - x^2}\,dx = \frac{49\pi}{2}$$



32. Semicircle

$$A = \tfrac{1}{2}\pi r^2$$

$$A = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = \tfrac{1}{2}\pi r^2$$



In Exercises 33 – 40, $\int_2^4 x^3\,dx = 60$, $\int_2^4 x\,dx = 6$, $\int_2^4 dx = 2$

33. $\int_4^2 x\,dx = -\int_2^4 x\,dx = -6$

34. $\int_2^2 x^3\,dx = 0$

35. $\int_2^4 8x\,dx = 8\int_2^4 x\,dx = 8(6) = 48$

36. $\int_2^4 25\,dx = 25\int_2^4 dx = 25(2) = 50$

37. $\int_2^4 (x - 9)\,dx = \int_2^4 x\,dx - 9\int_2^4 dx = 6 - 9(2) = -12$

38. $\int_2^4 (x^3 + 4)\,dx = \int_2^4 x^3\,dx + 4\int_2^4 dx = 60 + 4(2) = 68$

39. $\int_2^4 \left(\tfrac{1}{2}x^3 - 3x + 2\right)dx = \tfrac{1}{2}\int_2^4 x^3\,dx - 3\int_2^4 x\,dx + 2\int_2^4 dx$

$= \tfrac{1}{2}(60) - 3(6) + 2(2) = 16$

40. $\int_2^4 \left(10 + 4x - 3x^3\right)dx = 10\int_2^4 dx + 4\int_2^4 x\,dx - 3\int_2^4 x^3\,dx$

$= 10(2) + 4(6) - 3(60) = -136$

41. (a) $\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx = 10 + 3 = 13$

(b) $\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -10$

(c) $\int_5^5 f(x)\,dx = 0$

(d) $\int_0^5 3f(x)\,dx = 3\int_0^5 f(x)\,dx = 3(10) = 30$

42. (a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 4 + (-1) = 3$

(b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = -(-1) = 1$

(c) $\int_3^3 f(x)\,dx = 0$

(d) $\int_3^6 -5f(x)\,dx = -5\int_3^6 f(x)\,dx = -5(-1) = 5$

43. (a) $\int_2^6 [f(x) + g(x)]\,dx = \int_2^6 f(x)\,dx + \int_2^6 g(x)\,dx$

$= 10 + (-2) = 8$

(b) $\int_2^6 [g(x) - f(x)]\,dx = \int_2^6 g(x)\,dx - \int_2^6 f(x)\,dx$

$= -2 - 10 = -12$

(c) $\int_2^6 2g(x)\,dx = 2\int_2^6 g(x)\,dx = 2(-2) = -4$

(d) $\int_2^6 3f(x)\,dx = 3\int_2^6 f(x)\,dx = 3(10) = 30$

44. (a) $\int_{-1}^0 f(x)\,dx = \int_{-1}^1 f(x)\,dx - \int_0^1 f(x)\,dx$

$= 0 - 5 = -5$

(b) $\int_0^1 f(x)\,dx - \int_1^0 f(x)\,dx = 5 - (-5) = 10$

(c) $\int_{-1}^1 3f(x)\,dx = 3\int_{-1}^1 f(x)\,dx = 3(0) = 0$

(d) $\int_0^1 3f(x)\,dx = 3\int_0^1 f(x)\,dx = 3(5) = 15$

45. Lower estimate: $[24 + 12 - 4 - 20 - 36](2) = -48$

Upper estimate: $[32 + 24 + 12 - 4 - 20](2) = 88$

46. (a) $[-6 + 8 + 30](2) = 64$ left endpoint estimate

(b) $[8 + 30 + 80](2) = 236$ right endpoint estimate

(c) $[0 + 18 + 50](2) = 136$ midpoint estimate

If f is increasing, then (a) is below the actual value and (b) is above.

47. (a) Quarter circle below x-axis:

$-\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$

(b) Triangle: $\frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$

(c) Triangle + Semicircle below x-axis:

$-\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$

(d) Sum of parts (b) and (c): $4 - (1 + 2\pi) = 3 - 2\pi$

(e) Sum of absolute values of (b) and (c):
$4 + (1 + 2\pi) = 5 + 2\pi$

(f) Answers to (d) plus
$2(10) = 20$: $(3 - 2\pi) + 20 = 23 - 2\pi$

48. (a) $\int_0^1 -f(x)\,dx = -\int_0^1 f(x)\,dx = \frac{1}{2}$

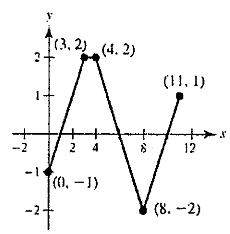
(b) $\int_3^4 3f(x)\,dx = 3(2) = 6$

(c) $\int_0^7 f(x)\,dx = -\frac{1}{2} + \frac{1}{2}(2)(2) + 2 + \frac{1}{2}(2)(2) - \frac{1}{2} = 5$

(d) $\int_5^{11} f(x)\,dx = \frac{1}{2} - \frac{1}{2}(4)(2) + \frac{1}{2} = -3$

(e) $\int_0^{11} f(x)\,dx = -\frac{1}{2} + 2 + 2 + 2 - 4 + \frac{1}{2} = 2$

(f) $\int_4^{10} f(x)\,dx = 2 - 4 = -2$



49. (a) $\int_0^5 [f(x) + 2]\,dx = \int_0^5 f(x)\,dx + \int_0^5 2\,dx$

$= 4 + 10 = 14$

(b) $\int_{-2}^3 f(x + 2)\,dx = \int_0^5 f(x)\,dx = 4$ (Let $u = x + 2$.)

(c) $\int_{-5}^5 f(x)\,dx = 2\int_0^5 f(x)\,dx = 2(4) = 8$ (f even)

(d) $\int_{-5}^5 f(x)\,dx = 0$ (f odd)

50. (a) The left endpoint approximation will be greater than the actual area so,

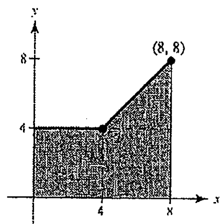
$$\sum_{i=1}^{n} f(x_i)\Delta x > \int_1^5 f(x)\,dx.$$

(b) The right endpoint approximation will be less than the actual area so,

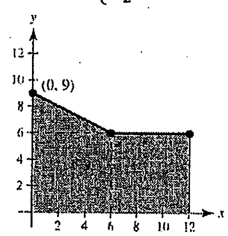
$$\sum_{i=1}^{n} f(x_i)\Delta x < \int_1^5 f(x)\,dx.$$

51. $f(x) = \begin{cases} 4, & x < 4 \\ x, & x \geq 4 \end{cases}$

$$\int_0^8 f(x)\,dx = 4(4) + 4(4) + \tfrac{1}{2}(4)(4) = 40$$

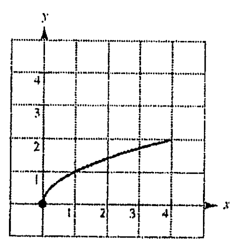


52. $f(x) = \begin{cases} 6, & x > 6 \\ -\frac{1}{2}x + 9, & x \leq 6 \end{cases}$



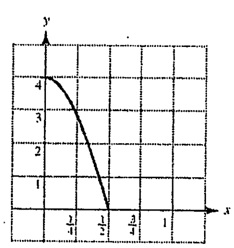
$$\int_0^{12} f(x)\,dx = 6(6) + \tfrac{1}{2}6(3) + 6(6) = 36 + 9 + 36 = 81$$

53.



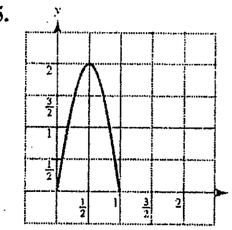
(a) $A \approx 5$ square units

54.



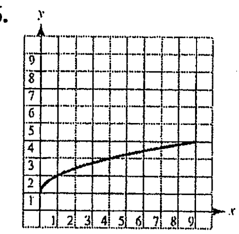
(b) $A \approx \frac{4}{3}$ square units

55.



(d) $A \approx \frac{5}{4}$ square units

56.



(c) $A \approx 27$ square units

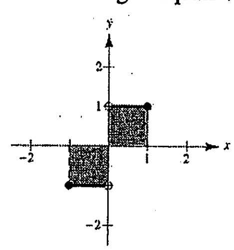
57. $f(x) = \dfrac{1}{x - 4}$

is not integrable on the interval [3, 5] because f has a discontinuity at $x = 4$.

58. $f(x) = |x|/x$ is integrable on $[-1, 1]$, but is not continuous on $[-1, 1]$. There is discontinuity at $x = 0$. To see that

$$\int_{-1}^{1} \frac{|x|}{x}\,dx$$

is integrable, sketch a graph of the region bounded by $f(x) = |x|/x$ and the x-axis for $-1 \leq x \leq 1$. You see that the integral equals 0.



59. $\int_{-2}^{1} f(x)\,dx + \int_{1}^{5} f(x)\,dx = \int_{-2}^{5} f(x)\,dx$

$a = -2, b = 5$

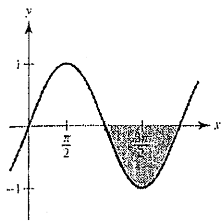
60. $\int_{-3}^{3} f(x)\,dx + \int_{3}^{6} f(x)\,dx - \int_{a}^{b} f(x)\,dx = \int_{-1}^{6} f(x)\,dx$

$$\int_{-3}^{6} f(x)\,dx + \int_{b}^{a} f(x)\,dx = \int_{-1}^{6} f(x)\,dx$$

$a = -3, b = -1$

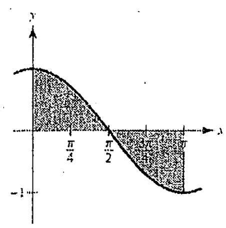
61. Answers will vary. *Sample answer:* $a = \pi$, $b = 2\pi$

$$\int_{\pi}^{2\pi} \sin x \, dx < 0$$



62. Answers will vary. *Sample answer:* $a = 0$, $b = \pi$

$$\int_{0}^{\pi} \cos x \, dx = 0$$



63. True

64. False

$$\int_0^1 x\sqrt{x}\, dx \neq \left(\int_0^1 x\, dx\right)\left(\int_0^1 \sqrt{x}\, dx\right)$$

65. True

66. True

67. False

$$\int_0^2 (-x)\, dx = -2$$

68. True. The limits of integration are the same.

69. $f(x) = x^2 + 3x, [0, 8]$

$x_0 = 0,\ x_1 = 1,\ x_2 = 3,\ x_3 = 7,\ x_4 = 8$

$\Delta x_1 = 1,\ \Delta x_2 = 2,\ \Delta x_3 = 4,\ \Delta x_4 = 1$

$c_1 = 1, c_2 = 2,\ c_3 = 5,\ c_4 = 8$

$$\sum_{i=1}^{4} f(c_i)\Delta x = f(1)\Delta x_1 + f(2)\Delta x_2 + f(5)\Delta x_3 + f(8)\Delta x_4$$
$$= (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272$$

70. $f(x) = \sin x, [0, 2\pi]$

$x_0 = 0,\ x_1 = \dfrac{\pi}{4},\ x_2 = \dfrac{\pi}{3},\ x_3 = \pi,\ x_4 = 2\pi$

$\Delta x_1 = \dfrac{\pi}{4},\ \Delta x_2 = \dfrac{\pi}{12},\ \Delta x_3 = \dfrac{2\pi}{3},\ \Delta x_4 = \pi$

$c_1 = \dfrac{\pi}{6},\ c_2 = \dfrac{\pi}{3},\ c_3 = \dfrac{2\pi}{3},\ c_4 = \dfrac{3\pi}{2}$

$$\sum_{i=1}^{4} f(c_i)\,\Delta x_i = f\left(\frac{\pi}{6}\right)\Delta x_1 + f\left(\frac{\pi}{3}\right)\Delta x_2 + f\left(\frac{2\pi}{3}\right)\Delta x_3 + f\left(\frac{3\pi}{2}\right)\Delta x_4$$
$$= \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{12}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2\pi}{3}\right) + (-1)(\pi) \approx -0.708$$

71. $\Delta x = \dfrac{b-a}{n},\ c_i = a + i(\Delta x) = a + i\left(\dfrac{b-a}{n}\right)$

$$\begin{aligned}
\int_0^b x\,dx &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[a + i\left(\frac{b-a}{n}\right)\right]\left(\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \left[\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} a + \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^{n} i\right] \\
&= \lim_{n\to\infty} \left[\frac{b-a}{n}(an) + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \left[a(b-a) + \frac{(b-a)^2}{n}\frac{n+1}{2}\right] \\
&= a(b-a) + \frac{(b-a)^2}{2} \\
&= (b-a)\left[a + \frac{b-a}{2}\right] \\
&= \frac{(b-a)(a+b)}{2} = \frac{b^2-a^2}{2}
\end{aligned}$$

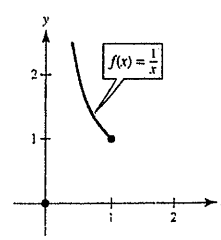
72. $\Delta x = \dfrac{b-a}{n},\ c_i = a + i(\Delta x) = a + i\left(\dfrac{b-a}{n}\right)$

$$\begin{aligned}
\int_a^b x^2\,dx &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[a + i\left(\frac{b-a}{n}\right)\right]^2\left(\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \left[\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} \left(a^2 + \frac{2ai(b-a)}{n} + i^2\left(\frac{b-a}{n}\right)^2\right)\right] \\
&= \lim_{n\to\infty} \left(\frac{b-a}{n}\right)\left[na^2 + \frac{2a(b-a)}{n}\frac{n(n+1)}{2} + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty} \left[a^2(b-a) + \frac{a(b-a)^2(n+1)}{n} + \frac{(b-a)^3}{6}\frac{(n+1)(2n+1)}{n^2}\right] \\
&= a^2(b-a) + a(b-a)^2 + \frac{1}{3}(b-a)^3 = \frac{1}{3}\left(b^3 - a^3\right)
\end{aligned}$$

73. $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

is not integrable on the interval $[0, 1]$. As $\|\Delta\| \to 0$, $f(c_i) = 1$ or $f(c_i) = 0$ in each subinterval because there are an infinite number of both rational and irrational numbers in any interval, no matter how small.

74. $f(x) = \begin{cases} 0, & x = 0 \\ \dfrac{1}{x}, & 0 < x \le 1 \end{cases}$



The limit

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

does not exist.

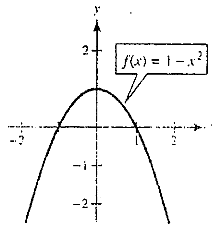
This does not contradict Theorem 4.4 because f is not continuous on $[0, 1]$.

75. The function f is nonnegative between $x = -1$ and $x = 1$.

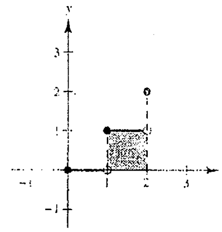
So, $\int_a^b (1 - x^2)\,dx$

is a maximum for

$a = -1$ and $b = 1$.



76. To find $\int_0^2 [\![x]\!]\,dx$, use a geometric approach.



So, $\int_0^2 [\![x]\!]\,dx = 1(2 - 1) = 1$.

77. Let $f(x) = x^2$, $0 \le x \le 1$, and $\Delta x_i = 1/n$. The appropriate Riemann Sum is

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

$$\lim_{n\to\infty}\frac{1}{n^3}\left[1^2 + 2^2 + 3^2 + \cdots + n^2\right] = \lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{n(2n+1)(n+1)}{6} = \lim_{n\to\infty}\frac{2n^2+3n+1}{6n^2} = \lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$

78. $I(f) - J(f) = \int_0^1 x^2 f(x)\,dx - \int_0^1 xf(x)^2\,dx.$

Observe that

$$\frac{x^3}{4} - x\left(f(x) - \frac{x}{2}\right)^2 = \frac{x^3}{4} - x\left(f(x)^2 - xf(x) + \frac{x^2}{4}\right) = \frac{x^3}{4} - xf(x)^2 + x^2 f(x) - \frac{x^3}{4} = x^2 f(x) - xf(x)^2$$

So, $I(f) - J(f) = \int_0^1\left[x^2 f(x) - xf(x)^2\right]dx = \int_0^1\left[\frac{x^3}{4} - x\left(f(x) - \frac{x}{2}\right)^2\right]dx \le \int_0^1 \frac{x^3}{4}\,dx = \frac{1}{16}$

Furthermore, $6 + f(x) = \frac{x}{2}$. Then $I(f) = \int_0^1 x^2\left(\frac{x}{2}\right)dx = \frac{1}{8}$ and $J(f) = \int_0^1 x\left(\frac{x^2}{4}\right) = \frac{1}{16}$

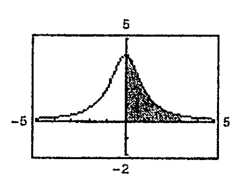
So $I(f) - J(f) = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}$

The maximum value is $\frac{1}{16}$.

Section 4.4 The Fundamental Theorem of Calculus

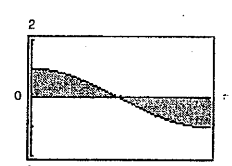
1. $f(x) = \dfrac{4}{x^2 + 1}$

$\int_0^{\pi} \frac{4}{x^2+1}\,dx$ is positive.



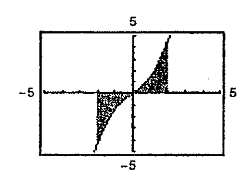
2. $f(x) = \cos x$

$\int_0^{\pi} \cos x\,dx = 0$



3. $f(x) = x\sqrt{x^2 + 1}$

$\int_{-2}^{2} x\sqrt{x^2+1}\,dx = 0$



4. $f(x) = x\sqrt{2 - x}$

$\int_{-2}^{2} x\sqrt{2 - x}\,dx$ is negative.