4.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Evaluating a Limit In Exercises 1 and 2, use Example 1 as a model to evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

over the region bounded by the graphs of the equations.

1. $f(x) = \sqrt{x}, \quad y = 0, \quad x = 0, \quad x = 3$

   $\left(\textit{Hint: } \text{Let } c_i = \dfrac{3i^2}{n^2}.\right)$

2. $f(x) = \sqrt[3]{x}, \quad y = 0, \quad x = 0, \quad x = 1$

   $\left(\textit{Hint: } \text{Let } c_i = \dfrac{i^3}{n^3}.\right)$

Evaluating a Definite Integral as a Limit In Exercises 3–8, evaluate the definite integral by the limit definition.

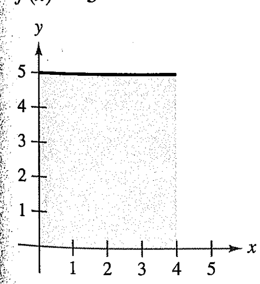
3. $\displaystyle\int_2^6 8\,dx$
4. $\displaystyle\int_{-2}^3 x\,dx$
5. $\displaystyle\int_{-1}^1 x^3\,dx$
6. $\displaystyle\int_1^4 4x^2\,dx$
7. $\displaystyle\int_1^2 (x^2 + 1)\,dx$
8. $\displaystyle\int_{-2}^1 (2x^2 + 3)\,dx$

Writing a Limit as a Definite Integral In Exercises 9–12, write the limit as a definite integral on the interval $[a, b]$, where c_i is any point in the ith subinterval.

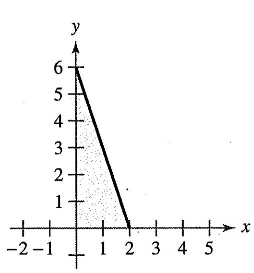
	Limit	Interval
9.	$\displaystyle\lim_{\lVert\Delta\rVert\to 0} \sum_{i=1}^{n} (3c_i + 10)\,\Delta x_i$	$[-1, 5]$
10.	$\displaystyle\lim_{\lVert\Delta\rVert\to 0} \sum_{i=1}^{n} 6c_i(4 - c_i)^2\,\Delta x_i$	$[0, 4]$
11.	$\displaystyle\lim_{\lVert\Delta\rVert\to 0} \sum_{i=1}^{n} \sqrt{c_i^2 + 4}\,\Delta x_i$	$[0, 3]$
12.	$\displaystyle\lim_{\lVert\Delta\rVert\to 0} \sum_{i=1}^{n} \left(\frac{3}{c_i^2}\right)\Delta x_i$	$[1, 3]$

Writing a Definite Integral In Exercises 13–22, set up a definite integral that yields the area of the region. (Do not evaluate the integral.)

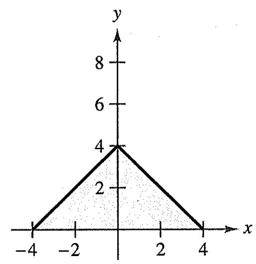
13. $f(x) = 5$



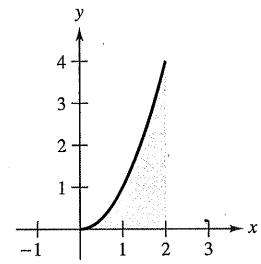
14. $f(x) = 6 - 3x$



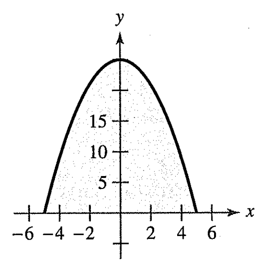
15. $f(x) = 4 - |x|$



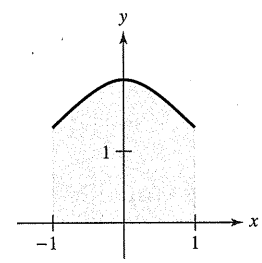
16. $f(x) = x^2$



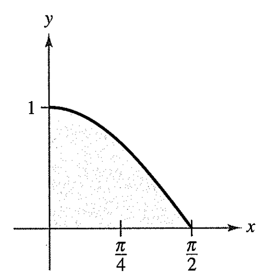
17. $f(x) = 25 - x^2$



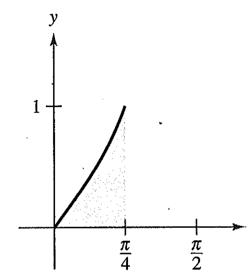
18. $f(x) = \dfrac{4}{x^2 + 2}$



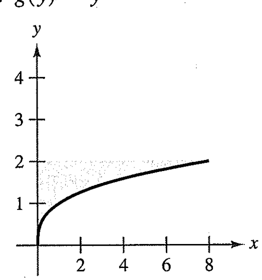
19. $f(x) = \cos x$



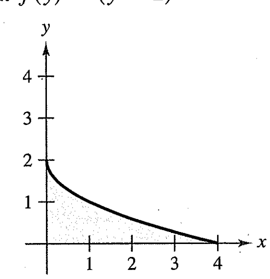
20. $f(x) = \tan x$



21. $g(y) = y^3$



22. $f(y) = (y - 2)^2$



Evaluating a Definite Integral Using a Geometric Formula In Exercises 23–32, sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral $(a > 0, r > 0)$.

23. $\displaystyle\int_0^3 4\,dx$
24. $\displaystyle\int_{-4}^6 6\,dx$
25. $\displaystyle\int_0^4 x\,dx$
26. $\displaystyle\int_0^8 \frac{x}{4}\,dx$

27. $\int_0^2 (3x + 4)\,dx$

28. $\int_0^3 (8 - 2x)\,dx$

29. $\int_{-1}^1 (1 - |x|)\,dx$

30. $\int_{-a}^a (a - |x|)\,dx$

31. $\int_{-7}^7 \sqrt{49 - x^2}\,dx$

32. $\int_{-r}^r \sqrt{r^2 - x^2}\,dx$

Using Properties of Definite Integrals **In Exercises 33–40, evaluate the integral using the following values.**

$$\int_2^4 x^3\,dx = 60, \qquad \int_2^4 x\,dx = 6, \qquad \int_2^4 dx = 2$$

33. $\int_4^2 x\,dx$

34. $\int_2^2 x^3\,dx$

35. $\int_2^4 8x\,dx$

36. $\int_2^4 25\,dx$

37. $\int_2^4 (x - 9)\,dx$

38. $\int_2^4 (x^3 + 4)\,dx$

39. $\int_2^4 \left(\frac{1}{2}x^3 - 3x + 2\right)dx$

40. $\int_2^4 (10 + 4x - 3x^3)\,dx$

41. **Using Properties of Definite Integrals** Given

$$\int_0^5 f(x)\,dx = 10 \quad \text{and} \quad \int_5^7 f(x)\,dx = 3$$

evaluate

(a) $\int_0^7 f(x)\,dx.$ (b) $\int_5^0 f(x)\,dx.$

(c) $\int_5^5 f(x)\,dx.$ (d) $\int_0^5 3f(x)\,dx.$

42. **Using Properties of Definite Integrals** Given

$$\int_0^3 f(x)\,dx = 4 \quad \text{and} \quad \int_3^6 f(x)\,dx = -1$$

evaluate

(a) $\int_0^6 f(x)\,dx.$ (b) $\int_6^3 f(x)\,dx.$

(c) $\int_3^3 f(x)\,dx.$ (d) $\int_3^6 -5f(x)\,dx.$

43. **Using Properties of Definite Integrals** Given

$$\int_2^6 f(x)\,dx = 10 \quad \text{and} \quad \int_2^6 g(x)\,dx = -2$$

evaluate

(a) $\int_2^6 [f(x) + g(x)]\,dx.$ (b) $\int_2^6 [g(x) - f(x)]\,dx.$

(c) $\int_2^6 2g(x)\,dx.$ (d) $\int_2^6 3f(x)\,dx.$

44. **Using Properties of Definite Integrals** Given

$$\int_{-1}^1 f(x)\,dx = 0 \quad \text{and} \quad \int_0^1 f(x)\,dx = 5$$

evaluate

(a) $\int_{-1}^0 f(x)\,dx.$ (b) $\int_0^1 f(x)\,dx - \int_{-1}^0 f(x)\,dx.$

(c) $\int_{-1}^1 3f(x)\,dx.$ (d) $\int_0^1 3f(x)\,dx.$

45. **Estimating a Definite Integral** Use the table of values to find lower and upper estimates of

$$\int_0^{10} f(x)\,dx.$$

Assume that f is a decreasing function.

x	0	2	4	6	8	10
$f(x)$	32	24	12	-4	-20	-36

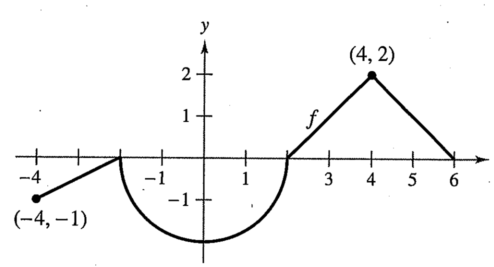
46. **Estimating a Definite Integral** Use the table of values to estimate

$$\int_0^6 f(x)\,dx.$$

Use three equal subintervals and the (a) left endpoints, (b) right endpoints, and (c) midpoints. When f is an increasing function, how does each estimate compare with the actual value? Explain your reasoning.

x	0	1	2	3	4	5	6
$f(x)$	-6	0	8	18	30	50	80

47. **Think About It** The graph of f consists of line segments and a semicircle, as shown in the figure. Evaluate each definite integral by using geometric formulas.

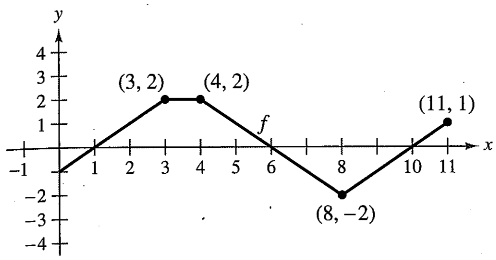


(a) $\int_0^2 f(x)\,dx$ (b) $\int_2^6 f(x)\,dx$

(c) $\int_{-4}^2 f(x)\,dx$ (d) $\int_{-4}^6 f(x)\,dx$

(e) $\int_{-4}^6 |f(x)|\,dx$ (f) $\int_{-4}^6 [f(x) + 2]\,dx$

48. Think About It The graph of f consists of line segments, as shown in the figure. Evaluate each definite integral by using geometric formulas.



(a) $\int_0^1 -f(x)\,dx$ (b) $\int_3^4 3f(x)\,dx$

(c) $\int_0^7 f(x)\,dx$ (d) $\int_5^{11} f(x)\,dx$

(e) $\int_0^{11} f(x)\,dx$ (f) $\int_4^{10} f(x)\,dx$

49. Think About It Consider the function f that is continuous on the interval $[-5, 5]$ and for which

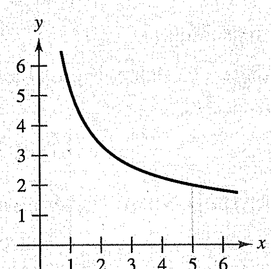
$$\int_0^5 f(x)\,dx = 4.$$

Evaluate each integral.

(a) $\int_0^5 [f(x) + 2]\,dx$ (b) $\int_{-2}^3 f(x + 2)\,dx$

(c) $\int_{-5}^5 f(x)\,dx$ (f is even.) (d) $\int_{-5}^5 f(x)\,dx$ (f is odd.)

50. HOW DO YOU SEE IT? Use the figure to fill in the blank with the symbol <, >, or =. Explain your reasoning.



(a) The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the left endpoint of the ith subinterval.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

(b) The interval $[1, 5]$ is partitioned into n subintervals of equal width Δx, and x_i is the right endpoint of the ith subinterval.

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \square \quad \int_1^5 f(x)\,dx$$

51. Think About It A function f is defined below. Use geometric formulas to find $\int_0^8 f(x)\,dx$.

$$f(x) = \begin{cases} 4, & x < 4 \\ x, & x \geq 4 \end{cases}$$

52. Think About It A function f is defined below. Use geometric formulas to find $\int_0^{12} f(x)\,dx$.

$$f(x) = \begin{cases} 6, & x > 6 \\ -\frac{1}{2}x + 9, & x \leq 6 \end{cases}$$

WRITING ABOUT CONCEPTS

Approximation In Exercises 53–56, determine which value best approximates the definite integral. Make your selection on the basis of a sketch.

53. $\int_0^4 \sqrt{x}\,dx$

(a) 5 (b) −3 (c) 10 (d) 2 (e) 8

54. $\int_0^{1/2} 4 \cos \pi x\,dx$

(a) 4 (b) $\frac{4}{3}$ (c) 16 (d) 2π (e) −6

55. $\int_0^1 2 \sin \pi x\,dx$

(a) 6 (b) $\frac{1}{2}$ (c) 4 (d) $\frac{5}{4}$

56. $\int_0^9 \left(1 + \sqrt{x}\right)dx$

(a) −3 (b) 9 (c) 27 (d) 3

57. **Determining Integrability** Determine whether the function

$$f(x) = \frac{1}{x - 4}$$

is integrable on the interval $[3, 5]$. Explain.

58. **Finding a Function** Give an example of a function that is integrable on the interval $[-1, 1]$, but not continuous on $[-1, 1]$.

Finding Values In Exercises 59–62, find possible values of a and b that make the statement true. If possible, use a graph to support your answer. (There may be more than one correct answer.)

59. $\int_{-2}^1 f(x)\,dx + \int_1^5 f(x)\,dx = \int_a^b f(x)\,dx$

60. $\int_{-3}^3 f(x)\,dx + \int_3^6 f(x)\,dx - \int_a^b f(x)\,dx = \int_{-1}^6 f(x)\,dx$

61. $\int_a^b \sin x\,dx < 0$

62. $\int_a^b \cos x\,dx = 0$

True or False? **In Exercises 63–68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

63. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

64. $\int_a^b f(x)g(x)\,dx = \left[\int_a^b f(x)\,dx\right]\left[\int_a^b g(x)\,dx\right]$

65. If the norm of a partition approaches zero, then the number of subintervals approaches infinity.

66. If f is increasing on $[a, b]$, then the minimum value of $f(x)$ on $[a, b]$ is $f(a)$.

67. The value of

$$\int_a^b f(x)\,dx$$

must be positive.

68. The value of

$$\int_2^2 \sin(x^2)\,dx$$

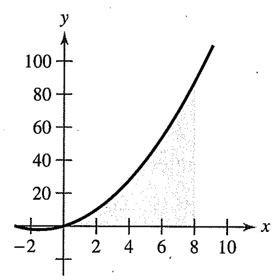
is 0.

69. Finding a Riemann Sum Find the Riemann sum for $f(x) = x^2 + 3x$ over the interval $[0, 8]$, where

$$x_0 = 0,\quad x_1 = 1,\quad x_2 = 3,\quad x_3 = 7,\quad \text{and} \quad x_4 = 8$$

and where

$$c_1 = 1,\quad c_2 = 2,\quad c_3 = 5,\quad \text{and} \quad c_4 = 8.$$

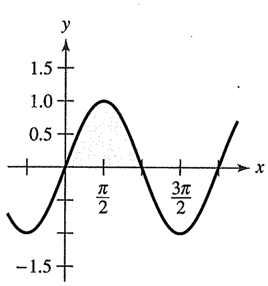


70. Finding a Riemann Sum Find the Riemann sum for $f(x) = \sin x$ over the interval $[0, 2\pi]$, where

$$x_0 = 0,\quad x_1 = \frac{\pi}{4},\quad x_2 = \frac{\pi}{3},\quad x_3 = \pi,\quad \text{and} \quad x_4 = 2\pi,$$

and where

$$c_1 = \frac{\pi}{6},\quad c_2 = \frac{\pi}{3},\quad c_3 = \frac{2\pi}{3},\quad \text{and} \quad c_4 = \frac{3\pi}{2}.$$



71. Proof Prove that $\int_a^b x\,dx = \frac{b^2 - a^2}{2}$.

72. Proof Prove that $\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}$.

73. Think About It Determine whether the Dirichlet function

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

is integrable on the interval $[0, 1]$. Explain.

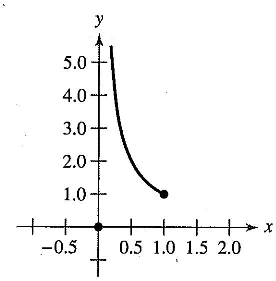
74. Finding a Definite Integral The function

$$f(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{x}, & 0 < x \le 1 \end{cases}$$

is defined on $[0, 1]$, as shown in the figure. Show that

$$\int_0^1 f(x)\,dx$$

does not exist. Why doesn't this contradict Theorem 4.4?



75. Finding Values Find the constants a and b that maximize the value of

$$\int_a^b (1 - x^2)\,dx.$$

Explain your reasoning.

76. Step Function Evaluate, if possible, the integral

$$\int_0^2 [\![x]\!]\,dx.$$

77. Using a Riemann Sum Determine

$$\lim_{n\to\infty} \frac{1}{n^3}[1^2 + 2^2 + 3^2 + \cdots + n^2]$$

by using an appropriate Riemann sum.

PUTNAM EXAM CHALLENGE

78. For each continuous function $f\colon [0, 1] \to \mathbb{R}$, let

$$I(f) = \int_0^1 x^2 f(x)\,dx \quad \text{and} \quad J(x) = \int_0^1 x(f(x))^2\,dx.$$

Find the maximum value of $I(f) - J(f)$ over all such functions f.

This problem was composed by the Committee on the Putnam Prize Competition.