# 4.4b Average Value Theorem (MVT for Integrals)

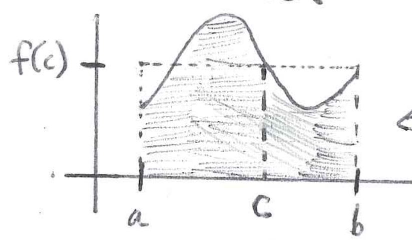
If f is able to be integrated on the closed interval [a, b], then the average value of f on the interval is:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Area

height

width




c-value exists on the closed interval [a, b]

* There exists a rectangle such that the area of the rectangle is the same as the area under the curve. f(c) is the height of rectangle.

Since (height)·(width) = Area, then $\text{height} = \frac{\text{Area}}{\text{width}}$

**Ex.1** ⓐ Find the average value of $f(x) = x^2+1$ on [2,5]

Use Avg. Value Theorem: $f(c) = \frac{1}{5-2}\int_2^5 x^2+1\,dx$

$$f(c) = \frac{1}{3}\cdot \left.\frac{x^3}{3}+x\right]_2^5 = \frac{1}{3}\left[\frac{125}{3}+5-\left(\frac{8}{3}+2\right)\right] = \frac{1}{3}\left[\frac{125}{3}-\frac{8}{3}+5-2\right]$$

$$= \frac{1}{3}\left[\frac{117}{3}+3\right] = \frac{1}{3}\left(\frac{126}{3}\right) = \boxed{14} \qquad \underline{f(c)=14}$$

ⓑ Find c-value.

$f(x) = x^2+1$

$f(c) = c^2+1$

$14 = c^2+1$

$13 = c^2$

$c = \pm\sqrt{13}$

$\underline{c = \sqrt{13}}$ since $2 < \sqrt{13} < 5$.

# 4.4b (continued) 2nd Fundamental Theorem of Calculus (SFTC)

## Definite Integral as a Function

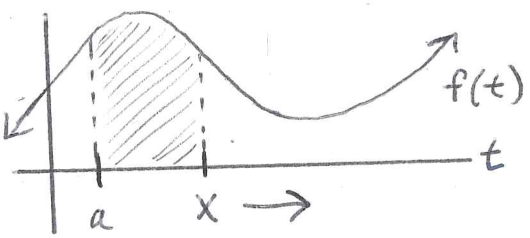
To recap, we've covered

1) (indefinite) general integrals (area-finding functions)
2) definite integrals (finds area between 2 x-values)

Now, there is also a function that is the integral itself.

Consider $f(x) = \int_a^x f(t)\,dt$

Instead of going from a constant to another constant, we are going from a constant to a moving value of x.




2nd Fundamental Theorem of Calculus (SFTC)

$$\frac{d}{dx}\left[\int_a^{p(x)} f(t)\,dt\right] = f(p(x)) \cdot p'(x)$$

* Very Important *

**Ex. 2**

a) Find $\frac{d}{dx}\left[\int_{-3}^{x} \sqrt{t^2+4}\,dt\right] = \sqrt{x^2+4} \cdot (1) = \boxed{\sqrt{x^2+4}}$

b) Find $\frac{d}{dx}\left[\int_{3}^{x^2} \sqrt{t-1}\,dt\right] = \sqrt{x^2-1} \cdot (2x) = 2x\sqrt{x^2-1}$

c) Find $\frac{d}{dx}\left[\int_{3x}^{0} \frac{1}{t+2}\,dt\right] = \frac{d}{dx}\left[-\int_{0}^{3x} \frac{1}{t+2}\,dt\right] = -\frac{1}{3x+2} \cdot 3 = \boxed{\frac{-3}{3x+2}}$

d) Find $\frac{d}{dx}\left[\int_{x}^{x^2} (2t+3)\,dt\right] \rightarrow \frac{d}{dx}\left[\frac{2t^2}{2} + 3t \Big|_x^{x^2}\right] =$

SFTC does not apply here. One of the bounds would need to be a constant

$= \frac{d}{dx}\left((x^2)^2 + 3x^2 - (x^2+3x)\right)$

$= \frac{d}{dx}\left(x^4 + 3x^2 - x^2 - 3x\right)$

$= \frac{d}{dx}\left(x^4 + 2x^2 - 3x\right)$

$= \boxed{4x^3 + 4x - 3}$

# $\boxed{4.4b}$ Avg. Value Formula SFTC

p. 291-293

#33-49 odd, 60, 75-91 odd

Avg. Value Theorem: $f(c) = \boxed{\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx}$

43) $f(x) = x - 2\sqrt{x}$ $[0,2]$ $\dfrac{1}{2-0}\displaystyle\int_0^2 x - 2\sqrt{x}\,dx$

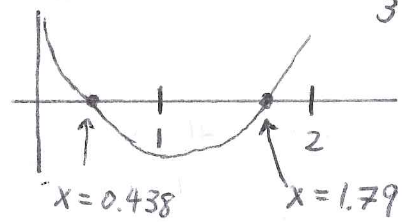
$f(c) = \dfrac{1}{2}\displaystyle\int_0^2 x - 2x^{1/2}\,dx = \dfrac{x^2}{2} - \dfrac{2x^{3/2}}{3/2} = \dfrac{x^2}{2} - \dfrac{2}{3}\left(2x^{3/2}\right) = \dfrac{x^2}{2} - \dfrac{4}{3}x^{3/2}\Big]_0^2 = \dfrac{2^2}{2} - \dfrac{4}{3}(2)^{3/2}$

$= 2 - \dfrac{4}{3}\sqrt{8}$

$f(c) = \dfrac{1}{2}\left(2 - \dfrac{4}{3}\sqrt{8}\right) = 1 - \dfrac{2}{3}\sqrt{8}$

$c - 2\sqrt{c} = 1 - \dfrac{2\sqrt{8}}{3} \longrightarrow c - 2\sqrt{c} - 1 + \dfrac{2\sqrt{8}}{3} = 0$

$\boxed{c = 0.438,\ 1.791}$




45) $f(x) = 2\sec^2 x$ $[-\pi/4, \pi/4]$ $f(c) = \dfrac{1}{\frac{\pi}{4} - \frac{-\pi}{4}}\displaystyle\int_{-\pi/4}^{\pi/4} 2\sec^2 x\,dx = \dfrac{1}{\pi/2}\displaystyle\int_{-\pi/4}^{\pi/4} 2\sec^2 x\,dx$

$f(c) = \dfrac{2}{\pi} \cdot 2\tan x\Big]_{-\pi/4}^{\pi/4} \rightarrow 2\tan\pi/4 - 2\tan(-\pi/4) = 2(1) - 2(-1) = 2 + 2 = 4$

$f(c) = \dfrac{2}{\pi}(4) = 8/\pi$ $f(c) = \dfrac{8}{\pi} \rightarrow 2\sec^2 x = \dfrac{8}{\pi}$ $\sec^2 x = \dfrac{4}{\pi}$ $\cos^2 x = \dfrac{\pi}{4}$

$\cos x = \pm\sqrt{\pi/4}$ $x = \pm\cos^{-1}\left(\sqrt{\pi/4}\right)$, $\boxed{x = \pm 0.4817}$

47) $f(x) = 4 - x^2$ $[-2, 2]$

$f(c) = \dfrac{1}{2--2}\displaystyle\int_{-2}^{2} 4 - x^2\,dx$

$f(c) = \dfrac{1}{4}\displaystyle\int_{-2}^{2} 4 - x^2\,dx$

$= \dfrac{1}{4}\ 4x - \dfrac{x^3}{3}\Big]_{-2}^{2}$

$= \dfrac{1}{4} \cdot 8 - 8/3 - (-8 + 8/3)$

$= \dfrac{1}{4} \cdot 16 - \dfrac{16}{3}$

$= \dfrac{1}{4}\left(\dfrac{32}{3}\right) = \dfrac{32}{12}\,4 = \dfrac{8}{3}$

$f(c) = 8/3$

$4 - c^2 = 8/3$

$-c^2 = 8/3 - 4 = -4/3$

$c^2 = 4/3$

$c = \pm\sqrt{4/3}$

$\boxed{c = \pm 1.155}$

Make sure c-values fall between intervals

4.4b (continued)

49) $f(x) = \sin x \ [0, \pi]$ $\quad f(c) = \frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx$

$f(c) = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx$

$= \frac{1}{\pi}\cdot(-\cos x)\Big]_0^{\pi}$

$= \frac{1}{\pi}\cdot -\cos\pi - (-\cos 0)$

$= \frac{1}{\pi}\cdot[-(-1)+1] = \frac{2}{\pi}$

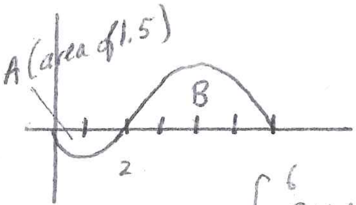
$\sin c = \frac{2}{\pi}$ ← 1st and 2nd quadrant!

$c = 0.69$, $c = \pi - 0.69$

$c = \sin^{-1}\left(\frac{2}{\pi}\right)$

$\boxed{c = 0.69,\ 2.451}$

60) Find avg. value of f over interval $[0,6]$




$\int_0^6 f(x)dx = 3.5$

$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$

$f(c) = \frac{1}{6-0}\int_0^6 f(x)dx$

$f(c) = \frac{1}{6}\cdot 3.5$

$= \frac{3.5}{6} = \boxed{\frac{7}{12}}$

#75-91 : SFTC: $\frac{d}{dx}\left[\int_0^x f(t)dt\right] = f(x)\cdot x'$

75) $F(x) = \int_0^x (t+2)dt$

a) $\int_0^x [t+2]dt = \frac{t^2}{2} + 2t\Big]_0^x = \frac{x^2}{2} + 2x$

b) $\frac{d}{dx}\left[\frac{x^2}{2} + 2x\right] = (x+2)$ $\quad = x+2$

83) $F(x) = \int_{-1}^x \sqrt{t^4+1}\,dt$

$\frac{d}{dx}\left[\int_{-1}^x (t^4+1)^{1/2}dt\right]$

$= (x^4+1)^{1/2}\cdot(1) = \sqrt{x^4+1}$

87) $F(x) = \int_x^{x+2} (4t+1)dt$

cannot use SFTC shortcut. Neither bound is 0.

$\frac{4t^2}{2} + t\Big]_x^{x+2}$

$= 2(x+2)^2 + x + 2 - (2x^2 + x)$

$2(x^2+4x+4) + x + 2 - 2x^2 - x$

$\cancel{2x^2} + 8x + 8 + \cancel{x} + 2 - \cancel{2x^2} - \cancel{x}$

$= \boxed{8x + 10}$

89) $F(x)=\int_0^{\sin x} \sqrt{t}\,dt$

$\frac{d}{dx}\left[\int_0^{\sin x} t^{1/2}\,dt\right]$

$= (\sin x)^{1/2} \cdot \cos x$

$= \boxed{\cos x \sqrt{\sin x}}$

91) $F(x)=\int_0^{x^3} \sin t^2\,dt$

$\frac{d}{dx}\left[\int_0^{x^2} \sin t^2\,dt\right]$

$= \sin (x^3)^2 \cdot 3x^2$

$= \boxed{3x^2 \sin x^6}$