4.5 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Finding u and du In Exercises 1–4, complete the table by identifying u and du for the integral.

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
1. $\int (8x^2 + 1)^2(16x)\,dx$		
2. $\int x^2\sqrt{x^3 + 1}\,dx$		
3. $\int \tan^2 x \sec^2 x\,dx$		
4. $\int \frac{\cos x}{\sin^2 x}\,dx$		

Finding an Indefinite Integral In Exercises 5–26, find the indefinite integral and check the result by differentiation.

5. $\int (1 + 6x)^4(6)\,dx$
6. $\int (x^2 - 9)^3(2x)\,dx$
7. $\int \sqrt{25 - x^2}\,(-2x)\,dx$
8. $\int \sqrt[3]{3 - 4x^2}(-8x)\,dx$
9. $\int x^3(x^4 + 3)^2\,dx$
10. $\int x^2(6 - x^3)^5\,dx$
11. $\int x^2(x^3 - 1)^4\,dx$
12. $\int x(5x^2 + 4)^3\,dx$
13. $\int t\sqrt{t^2 + 2}\,dt$
14. $\int t^3\sqrt{2t^4 + 3}\,dt$
15. $\int 5x\,\sqrt[3]{1 - x^2}\,dx$
16. $\int u^2\sqrt{u^3 + 2}\,du$
17. $\int \frac{x}{(1 - x^2)^3}\,dx$
18. $\int \frac{x^3}{(1 + x^4)^2}\,dx$
19. $\int \frac{x^2}{(1 + x^3)^2}\,dx$
20. $\int \frac{6x^2}{(4x^3 - 9)^3}\,dx$
21. $\int \frac{x}{\sqrt{1 - x^2}}\,dx$
22. $\int \frac{x^3}{\sqrt{1 + x^4}}\,dx$
23. $\int \left(1 + \frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt$
24. $\int \left[x^2 + \frac{1}{(3x)^2}\right]dx$
25. $\int \frac{1}{\sqrt{2x}}\,dx$
26. $\int \frac{x}{\sqrt[3]{5x^2}}\,dx$

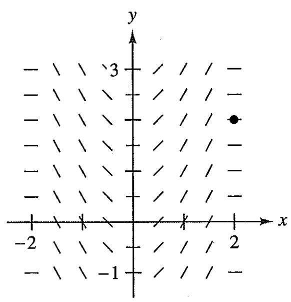
Differential Equation In Exercises 27–30, solve the differential equation.

27. $\frac{dy}{dx} = 4x + \frac{4x}{\sqrt{16 - x^2}}$
28. $\frac{dy}{dx} = \frac{10x^2}{\sqrt{1 + x^3}}$
29. $\frac{dy}{dx} = \frac{x + 1}{(x^2 + 2x - 3)^2}$
30. $\frac{dy}{dx} = \frac{x - 4}{\sqrt{x^2 - 8x + 1}}$

Slope Field In Exercises 31 and 32, a differential equation, a point, and a slope field are given. A *slope field* consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the directions of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (To print an enlarged copy of the graph, go to *MathGraphs.com*.) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).

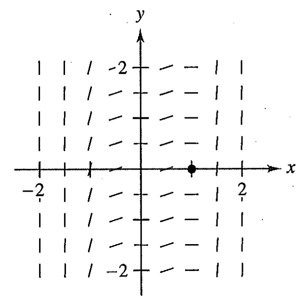
31. $\frac{dy}{dx} = x\sqrt{4 - x^2}$

$(2, 2)$



32. $\frac{dy}{dx} = x^2(x^3 - 1)^2$

$(1, 0)$



Finding an Indefinite Integral In Exercises 33–42, find the indefinite integral.

33. $\int \pi \sin \pi x\,dx$
34. $\int \sin 4x\,dx$
35. $\int \cos 8x\,dx$
36. $\int \csc^2\left(\frac{x}{2}\right)dx$
37. $\int \frac{1}{\theta^2}\cos\frac{1}{\theta}\,d\theta$
38. $\int x \sin x^2\,dx$
39. $\int \sin 2x \cos 2x\,dx$
40. $\int \sqrt{\tan x}\,\sec^2 x\,dx$
41. $\int \frac{\csc^2 x}{\cot^3 x}\,dx$
42. $\int \frac{\sin x}{\cos^3 x}\,dx$

Finding an Equation In Exercises 43–46, find an equation for the function f that has the given derivative and whose graph passes through the given point.

Derivative	Point
43. $f'(x) = -\sin\frac{x}{2}$	$(0, 6)$
44. $f'(x) = \sec^2(2x)$	$\left(\frac{\pi}{2}, 2\right)$
45. $f'(x) = 2x(4x^2 - 10)^2$	$(2, 10)$
46. $f'(x) = -2x\sqrt{8 - x^2}$	$(2, 7)$

Change of Variables **In Exercises 47–54, find the indefinite integral by the method shown in Example 5.**

47. $\int x\sqrt{x+6}\,dx$

48. $\int x\sqrt{3x-4}\,dx$

49. $\int x^2\sqrt{1-x}\,dx$

50. $\int (x+1)\sqrt{2-x}\,dx$

51. $\int \frac{x^2-1}{\sqrt{2x-1}}\,dx$

52. $\int \frac{2x+1}{\sqrt{x+4}}\,dx$

53. $\int \frac{-x}{(x+1)-\sqrt{x+1}}\,dx$

54. $\int t\sqrt[3]{t+10}\,dt$

Evaluating a Definite Integral **In Exercises 55–62, evaluate the definite integral. Use a graphing utility to verify your result.**

55. $\int_{-1}^{1} x(x^2+1)^3\,dx$

56. $\int_{0}^{1} x^3(2x^4+1)^2\,dx$

57. $\int_{1}^{2} 2x^2\sqrt{x^3+1}\,dx$

58. $\int_{0}^{1} x\sqrt{1-x^2}\,dx$

59. $\int_{0}^{4} \frac{1}{\sqrt{2x+1}}\,dx$

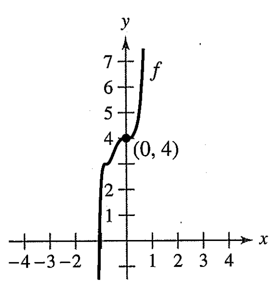
60. $\int_{0}^{2} \frac{x}{\sqrt{1+2x^2}}\,dx$

61. $\int_{1}^{9} \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}\,dx$

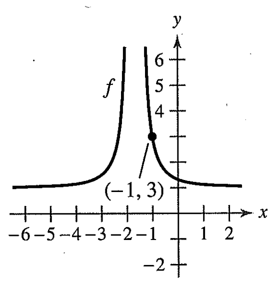
62. $\int_{1}^{5} \frac{x}{\sqrt{2x-1}}\,dx$

Differential Equation **In Exercises 63 and 64, the graph of a function f is shown. Use the differential equation and the given point to find an equation of the function.**

63. $\frac{dy}{dx} = 18x^2(2x^3+1)^2$

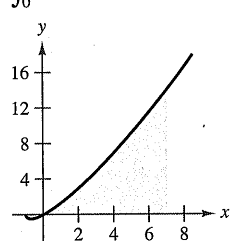


64. $\frac{dy}{dx} = \frac{-48}{(3x+5)^3}$

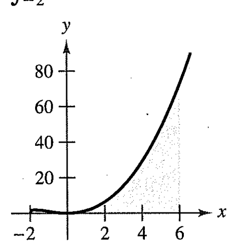


Finding the Area of a Region **In Exercises 65–68, find the area of the region. Use a graphing utility to verify your result.**

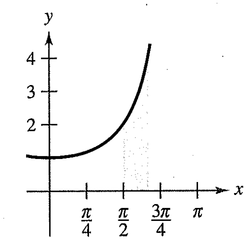
65. $\int_{0}^{7} x\sqrt[3]{x+1}\,dx$



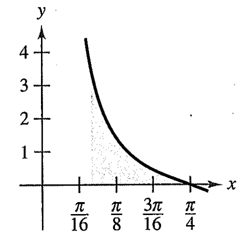
66. $\int_{-2}^{6} x^2\sqrt[3]{x+2}\,dx$



67. $\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)dx$



68. $\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x\,dx$



Even and Odd Functions **In Exercises 69–72, evaluate the integral using the properties of even and odd functions as an aid.**

69. $\int_{-2}^{2} x^2(x^2+1)\,dx$

70. $\int_{-2}^{2} x(x^2+1)^3\,dx$

71. $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\,dx$

72. $\int_{-\pi/2}^{\pi/2} \sin x \cos x\,dx$

73. Using an Even Function Use $\int_0^4 x^2\,dx = \frac{64}{3}$ to evaluate each definite integral without using the Fundamental Theorem of Calculus.

(a) $\int_{-4}^{0} x^2\,dx$ (b) $\int_{-4}^{4} x^2\,dx$

(c) $\int_{0}^{4} -x^2\,dx$ (d) $\int_{-4}^{0} 3x^2\,dx$

74. Using Symmetry Use the symmetry of the graphs of the sine and cosine functions as an aid in evaluating each definite integral.

(a) $\int_{-\pi/4}^{\pi/4} \sin x\,dx$ (b) $\int_{-\pi/4}^{\pi/4} \cos x\,dx$

(c) $\int_{-\pi/2}^{\pi/2} \cos x\,dx$ (d) $\int_{-\pi/2}^{\pi/2} \sin x \cos x\,dx$

Even and Odd Functions **In Exercises 75 and 76, write the integral as the sum of the integral of an odd function and the integral of an even function. Use this simplification to evaluate the integral.**

75. $\int_{-3}^{3} (x^3+4x^2-3x-6)\,dx$

76. $\int_{-\pi/2}^{\pi/2} (\sin 4x + \cos 4x)\,dx$

WRITING ABOUT CONCEPTS

77. Using Substitution Describe why

$$\int x(5-x^2)^3\,dx \neq \int u^3\,du$$

where $u = 5 - x^2$.

78. Analyzing the Integrand Without integrating, explain why

$$\int_{-2}^{2} x(x^2+1)^2\,dx = 0.$$

WRITING ABOUT CONCEPTS (continued)

79. Choosing an Integral You are asked to find one of the integrals. Which one would you choose? Explain.

(a) $\int \sqrt{x^3 + 1}\, dx$ or $\int x^2\sqrt{x^3 + 1}\, dx$

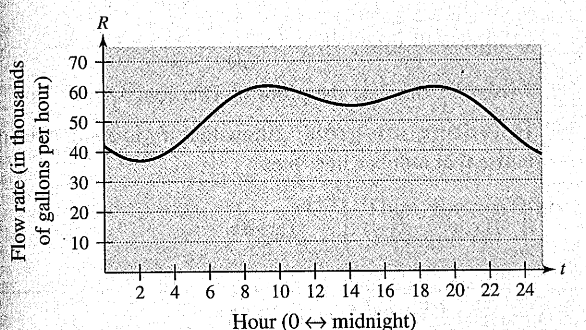
(b) $\int \tan(3x)\sec^2(3x)\, dx$ or $\int \tan(3x)\, dx$

80. Comparing Methods Find the indefinite integral in two ways. Explain any difference in the forms of the answers.

(a) $\int (2x - 1)^2\, dx$ (b) $\int \tan x \sec^2 x\, dx$

81. Depreciation The rate of depreciation dV/dt of a machine is inversely proportional to the square of $(t + 1)$, where V is the value of the machine t years after it was purchased. The initial value of the machine was \$500,000, and its value decreased \$100,000 in the first year. Estimate its value after 4 years.

82. HOW DO YOU SEE IT? The graph shows the flow rate of water at a pumping station for one day.



(a) Approximate the maximum flow rate at the pumping station. At what time does this occur?

(b) Explain how you can find the amount of water used during the day.

(c) Approximate the two-hour period when the least amount of water is used. Explain your reasoning.

83. Sales The sales S (in thousands of units) of a seasonal product are given by the model

$$S = 74.50 + 43.75 \sin \frac{\pi t}{6}$$

where t is the time in months, with $t = 1$ corresponding to January. Find the average sales for each time period.

(a) The first quarter $(0 \le t \le 3)$

(b) The second quarter $(3 \le t \le 6)$

(c) The entire year $(0 \le t \le 12)$

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84. Electricity

The oscillating current in an electrical circuit is

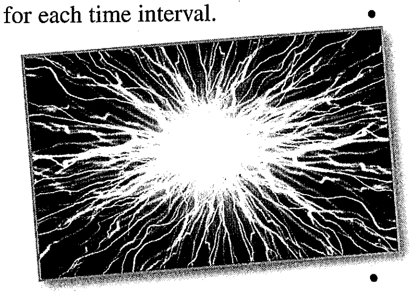
$I = 2\sin(60\pi t) + \cos(120\pi t)$

where I is measured in amperes and t is measured in seconds. Find the average current for each time interval.

(a) $0 \le t \le \frac{1}{60}$

(b) $0 \le t \le \frac{1}{240}$

(c) $0 \le t \le \frac{1}{30}$



Probability In Exercises 85 and 86, the function

$$f(x) = kx^n(1 - x)^m, \quad 0 \le x \le 1$$

where $n > 0$, $m > 0$, and k is a constant, can be used to represent various probability distributions. If k is chosen such that

$$\int_0^1 f(x)\, dx = 1$$

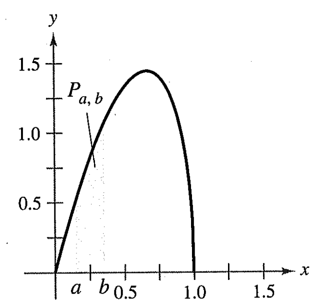
then the probability that x will fall between a and b $(0 \le a \le b \le 1)$ is

$$P_{a,b} = \int_a^b f(x)\, dx.$$

85. The probability that a person will remember between $100a\%$ and $100b\%$ of material learned in an experiment is

$$P_{a,b} = \int_a^b \frac{15}{4}x\sqrt{1 - x}\, dx$$

where x represents the proportion remembered. (See figure.)



(a) For a randomly chosen individual, what is the probability that he or she will recall between 50% and 75% of the material?

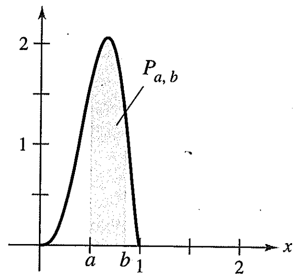
(b) What is the median percent recall? That is, for what value of b is it true that the probability of recalling 0 to b is 0.5?

86. The probability that ore samples taken from a region contain between $100a\%$ and $100b\%$ iron is

$$P_{a,\,b} = \int_a^b \frac{1155}{32} x^3(1 - x)^{3/2}\,dx$$

where x represents the proportion of iron. (See figure.) What is the probability that a sample will contain between

(a) 0% and 25% iron? (b) 50% and 100% iron?



87. Graphical Analysis Consider the functions f and g, where

$$f(x) = 6 \sin x \cos^2 x \quad \text{and} \quad g(t) = \int_0^t f(x)\,dx.$$

(a) Use a graphing utility to graph f and g in the same viewing window.

(b) Explain why g is nonnegative.

(c) Identify the points on the graph of g that correspond to the extrema of f.

(d) Does each of the zeros of f correspond to an extremum of g? Explain.

(e) Consider the function

$$h(t) = \int_{\pi/2}^t f(x)\,dx.$$

Use a graphing utility to graph h. What is the relationship between g and h? Verify your conjecture.

88. Finding a Limit Using a Definite Integral Find

$$\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{\sin(i\pi/n)}{n}$$

by evaluating an appropriate definite integral over the interval $[0, 1]$.

89. Rewriting Integrals

(a) Show that $\displaystyle\int_0^1 x^2(1 - x)^5\,dx = \int_0^1 x^5(1 - x)^2\,dx.$

(b) Show that $\displaystyle\int_0^1 x^a(1 - x)^b\,dx = \int_0^1 x^b(1 - x)^a\,dx.$

90. Rewriting Integrals

(a) Show that $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx.$

(b) Show that $\displaystyle\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$, where n is a positive integer.

True or False? **In Exercises 91–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. $\displaystyle\int (2x + 1)^2\,dx = \tfrac{1}{3}(2x + 1)^3 + C$

92. $\displaystyle\int x(x^2 + 1)\,dx = \tfrac{1}{2}x^2\left(\tfrac{1}{3}x^3 + x\right) + C$

93. $\displaystyle\int_{-10}^{10} (ax^3 + bx^2 + cx + d)\,dx = 2\int_0^{10} (bx^2 + d)\,dx$

94. $\displaystyle\int_a^b \sin x\,dx = \int_a^{b+2\pi} \sin x\,dx$

95. $\displaystyle 4\int \sin x \cos x\,dx = -\cos 2x + C$

96. $\displaystyle\int \sin^2 2x \cos 2x\,dx = \tfrac{1}{3}\sin^3 2x + C$

97. Rewriting Integrals Assume that f is continuous everywhere and that c is a constant. Show that

$$\int_{ca}^{cb} f(x)\,dx = c\int_a^b f(cx)\,dx.$$

98. Integration and Differentiation

(a) Verify that $\sin u - u\cos u + C = \displaystyle\int u \sin u\,du.$

(b) Use part (a) to show that $\displaystyle\int_0^{\pi^2} \sin\sqrt{x}\,dx = 2\pi.$

99. Proof Complete the proof of Theorem 4.16.

100. Rewriting Integrals Show that if f is continuous on the entire real number line, then

$$\int_a^b f(x + h)\,dx = \int_{a+h}^{b+h} f(x)\,dx.$$

PUTNAM EXAM CHALLENGE

101. If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n + 1} = 0,$$

show that the equation

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

has at least one real root.

102. Find all the continuous positive functions $f(x)$, for $0 \le x \le 1$, such that

$$\int_0^1 f(x)\,dx = 1$$

$$\int_0^1 f(x)x\,dx = \alpha$$

$$\int_0^1 f(x)x^2\,dx = \alpha^2$$

where α is a given real number.