Calculus **Chapter 4.5a** **U-Substitution Method for Indefinite Integrals**

U-Substitution is the reverse of the derivative chain rule. We want to look for a function and its derivative to be in the integral.

Example 1:

$Suppose\ f(x) = \sin(3x)$

$f'(x) = \cos(3x) \cdot 3$ $f'(x) = 3\cos(3x)$	This means that: $\int 3\cos(3x)dx = \sin(3x) + C$

U-Substitution Steps:

1. Assign the 'u' value to the expression inside the parentheses.
2. Find the derivative of u: $\frac{du}{dx}$
3. Solve for dx.
4. Rewrite the integral in terms of u and du. (Check to make sure no 'x' or 'dx' remains)
5. Evaluate the Integral
6. Write the answer in terms of x.
7. **Not all Integral problems require U-Substitution. Check first to see if expansion/rewriting problem will allow problem to only need the Power Rule**

Ex. 2: $\int x(x^2+1)^{15}dx$

Ex. 3: $\int x^2 \sec^2(2x^3)dx$

Ex. 4: $\int x^3\sqrt{5-x^4}\,dx$

Ex. 5: $\int \tan^5 x \sec^2 x \, dx$

Ex. 6: $\int (3-y)\left(\frac{1}{\sqrt{y}}\right) dy$

Change of Variable U-Substitution Method:

Ex. 7: $\int x\sqrt{x+3}\,dx$

Ex. 8: $\int x^2\sqrt{2-x}\,dx$

Calculus **Chapter 4.5a** **U-Substitution Method for Indefinite Integrals**

Key

U-Substitution is the reverse of the derivative chain rule. We want to look for a function and its derivative to be in the integral.

*U-substitution is a method of rewriting an integral problem into a simpler one to help us identify an Integral Rule appropriate for the problem.

Example 1:

Suppose $f(x) = \sin(3x)$

$f'(x) = \cos(3x) \cdot 3$
$f'(x) = 3\cos(3x)$

This means that:

$\int 3\cos(3x)dx = \sin(3x) + C$

U-Substitution Steps:

1. Assign the 'u' value to the expression inside the parentheses.
2. Find the derivative of u: $\frac{du}{dx}$
3. Solve for dx.
4. Rewrite the integral in terms of u and du. (Check to make sure no 'x' or 'dx' remains)
5. Evaluate the Integral
6. Write the answer in terms of x.
7. **Not all Integral problems require U-Substitution. Check first to see if expansion/rewriting problem will allow problem to only need the Power Rule**

Ex. 2: $\int x(x^2+1)^{15}dx$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int x \cdot u^{15} \cdot \frac{du}{2x}$

$\int \cancel{x} \cdot u^{15} \cdot \frac{du}{2\cancel{x}}$

$\frac{1}{2}\int u^{15}du = \frac{1}{2} \cdot \frac{u^{16}}{16} + C$

$= \boxed{\frac{1}{32}(x^2+1)^{16} + C}$

Be sure that variable 'x's cancel out. Remaining constants, coefficients are ok.

Ex. 3: $\int x^2 \sec^2(2x^3)dx$

$u = 2x^3$

$\frac{du}{dx} = 6x^2$

$dx = \frac{du}{6x^2}$

$\int \cancel{x^2} \cdot \sec^2 u \cdot \frac{du}{6\cancel{x^2}}$

$\frac{1}{6}\int \sec^2 u\, du$

$= \frac{1}{6}\tan u + C$

$= \boxed{\frac{1}{6}\tan(2x^3) + C}$

Ex. 4: $\int x^3\sqrt{5-x^4}\,dx = \int x^3(5-x^4)^{1/2}dx$

$u = 5-x^4$

$\frac{du}{dx} = -4x^3$

$dx = \frac{du}{-4x^3}$

$\int \cancel{x^3} \cdot u^{1/2} \cdot \frac{du}{-4\cancel{x^3}}$

$\frac{-1}{4}\int u^{1/2}du$

$= \frac{-1}{4}\frac{u^{3/2}}{3/2} + C$

$= \boxed{-\frac{1}{6}(5-x^4)^{3/2} + C}$

Ex. 5: $\int \tan^5 x \sec^2 x dx$

$$\int (\tan x)^5 (\sec x)^2 dx$$

$u = \tan x$ | $dx = \frac{du}{\sec^2 x}$

$\frac{du}{dx} = \sec^2 x$

$$\int (u)^5 \cdot \cancel{\sec^2 x} \cdot \frac{du}{\cancel{\sec^2 x}} = \int u^5 du$$

$$= \frac{u^6}{6} + c = \boxed{\frac{1}{6}\tan^6 x + C}$$

Ex. 6: $\int (3-y)\left(\frac{1}{\sqrt{y}}\right)dy$

$$\int (3-y)(y^{-1/2})dy$$

$$\int 3y^{-1/2} - y^{1/2} dy$$

$$\frac{3y^{1/2}}{1/2} - \frac{y^{3/2}}{3/2} + c$$

$$\boxed{6y^{1/2} - \frac{2}{3}y^{3/2} + C}$$

Change of Variable U-Substitution Method:

Ex. 7: $\int x\sqrt{x+3}dx$

$$\int x(x+3)^{1/2} dx$$

$u = x+3$

$\frac{du}{dx} = 1$

$dx = du$

*Creative method of substitution in order to eliminate x-variable

$$\int x \cdot u^{1/2} du$$

$$x = u - 3$$

$$\int (u-3)u^{1/2} du$$

$$\int u^{3/2} - 3u^{1/2} du$$

$$\frac{u^{5/2}}{5/2} - \frac{3u^{3/2}}{3/2} + c$$

$$\boxed{\frac{2}{5}(x+3)^{5/2} - 2(x+3)^{3/2} + C}$$

Ex. 8: $\int x^2\sqrt{2-x}dx$

$$\int x^2(2-x)^{1/2} dx$$

$u = 2-x$

$\frac{du}{dx} = -1$

$dx = -du$

$$\int x^2 \cdot u^{1/2} \cdot (-du)$$

$$x = 2-u$$

$$-\int (2-u)^2 u^{1/2} du$$

$$-\int (4-4u+u^2)u^{1/2} du$$

$$= \int -4u^{1/2} + 4u^{3/2} - u^{5/2}$$

$$= -\frac{4u^{3/2}}{3/2} + \frac{4u^{5/2}}{5/2} - \frac{u^{7/2}}{7/2} + c$$

$$\boxed{-\frac{8}{3}(2-x)^{3/2} + \frac{8}{5}(2-x)^{5/2} - \frac{2}{7}(2-x)^{7/2} + C}$$