4.5a U-Substitution (Indefinite Integrals)

Ex. 1 Suppose $f(x) = \sin(3x)$

$f'(x) = \cos 3x \cdot 3$

$f'(x) = 3\cos 3x$

This means that:

$\int 3\cos(3x)\,dx = \sin 3x + C$

U-Substitution method:

- Reverse of chain rule
- look for function and its derivative in the integral

Steps:

1) Assign "u" to the inside function (value inside parentheses)
2) Find derivative of u : $\frac{du}{dx}$
3) Solve for dx
4) Rewrite integral in terms of u and du (Be sure that no "x" and "dx" remain)
5) Evaluate Integral
6) Present answer in terms of x.

Ex. 2 $\int x(x^2+1)^{15}\,dx$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$2x\,dx = du$

$dx = \frac{du}{2x}$

$\int \cancel{x} \cdot u^{15} \cdot \frac{du}{2\cancel{x}}$

Be sure all "x" cancel out. If not, re-examine problem or use another method.

$= \int u^{15} \cdot \frac{du}{2}$

Include remaining constants in solution (It's ok to have remaining constants)

$= \frac{1}{2}\int u^{15}\,du$

$= \frac{1}{2} \cdot \frac{u^{16}}{16} + C$

$= \frac{1}{32}u^{16} + C$

$= \boxed{\frac{1}{32}(x^2+1)^{16} + C}$

Ex. 3 $\int x^2 \sec^2(2x^3)\,dx$

$u = 2x^3$

$\frac{du}{dx} = 6x^2$

$6x^2\,dx = du$

$dx = \frac{du}{6x^2}$

$\int \cancel{x^2} \cdot \sec^2 u \cdot \frac{du}{6\cancel{x^2}}$

$= \frac{1}{6}\int \sec^2 u\,du$

$= \frac{1}{6}\tan u + C$

$= \boxed{\frac{1}{6}\tan(2x^3) + C}$

4.5a (continued)

Ex. 4 $\int x^3\sqrt{5-x^4}\,dx$

$\int x^3(5-x^4)^{1/2}dx$

$u=5-x^4$

$\frac{du}{dx}=-4x^3$

$-4x^3dx=du$

$dx=\frac{du}{-4x^3}$

$\int \cancel{x^3}\cdot u^{1/2}\cdot\frac{du}{-4\cancel{x^3}}$

$-\frac{1}{4}\int u^{1/2}du$

$-\frac{1}{4}\ \frac{u^{3/2}}{3/2}+C$

$=-\frac{1}{4}\cdot\frac{2}{3}u^{3/2}+C$

$=\boxed{-\frac{1}{6}(5-x^4)^{3/2}+C}$

or $-\frac{1}{6}\sqrt{(5-x^4)^3}+C$

Ex. 5 $\int \tan^5x\sec^2x\,dx$

$\int(\tan x)^5(\sec x)^2dx$

$u=\tan x$

$\frac{du}{dx}=\sec^2x$

$dx=\frac{du}{\sec^2x}$

$\int u^5\cdot\cancel{\sec^2x}\cdot\frac{du}{\cancel{\sec^2x}}$

$\int u^5du$

$=\frac{u^6}{6}+C$

$\frac{1}{6}(\tan x)^6+C$

$=\boxed{\frac{1}{6}\tan^6x+C}$

Change of Variable U-substitution method

Ex. 6 $\int x\sqrt{x+3}\,dx$

$\int x(x+3)^{1/2}dx$

$u=x+3$

$\frac{du}{dx}=1$

$dx=du$

$\int x\cdot u^{1/2}du$

x does not cancel out. We can work around this by replacing x in terms of u.

$x=u-3$

$\int(u-3)u^{1/2}du$

$\int u^{3/2}-3u^{1/2}du$

$=\frac{u^{5/2}}{5/2}-\frac{3u^{3/2}}{3/2}+C$

$=\frac{2}{5}u^{5/2}-\cancel{3}\cdot\frac{2}{\cancel{3}}u^{3/2}+C$

$=\boxed{\frac{2}{5}\sqrt{(x+3)^5}-2\sqrt{(x+3)^3}+C}$

$\boxed{4.5a}$ U-substitution p. 304-305 #7-33 odd, 43-69 odd

9) $\int \sqrt{9-x^2}\,(-2x)\,dx$ | $u = 9-x^2$, $\frac{du}{dx} = -2x$, $dx = \frac{du}{-2x}$ | $= \int u^{1/2} \cdot \cancel{-2x} \cdot \frac{du}{\cancel{-2x}}$ | $= \frac{u^{3/2}}{3/2} + C$

$= \int (9-x^2)^{1/2}(-2x\,dx)$ | $= \int u^{1/2}\,du$ | $= \frac{2}{3}u^{3/2} + C$

$= \boxed{\frac{2}{3}(9-x^2)^{3/2} + C}$

13) $\int x^2(x^3-1)^4\,dx$ | $\int \cancel{x^2} \cdot u^4 \cdot \frac{du}{3\cancel{x^2}}$ | $= \frac{1}{15}u^5 + C$

$u = x^3 - 1$ | $\frac{1}{3}\int u^4\,du$ | $= \boxed{\frac{1}{15}(x^3-1)^5 + C}$

$\frac{du}{dx} = 3x^2$ | $\frac{1}{3} \cdot \frac{u^5}{5} + C$

$dx = \frac{du}{3x^2}$

17) $\int 5x\sqrt[3]{1-x^2}\,dx$ | $\int 5\cancel{x} \cdot u^{1/3} \cdot \frac{du}{-2\cancel{x}}$ | $= \frac{-5}{2} \cdot \frac{3}{4}u^{4/3} + C$

$= -15/8\, u^{4/3} + C$

$\int 5x(1-x^2)^{1/3}\,dx$ | $\frac{-5}{2}\int u^{1/3}\,du$ | $= \boxed{\frac{-15}{8}(1-x^2)^{4/3} + C}$

$u = 1-x^2$, $dx = \frac{du}{-2x}$ | $\frac{-5}{2}\left(\frac{u^{4/3}}{4/3}\right) + C$

$\frac{du}{dx} = -2x$

21) $\int \frac{x^2}{(1+x^3)^2}\,dx$ | $= \int \frac{\cancel{x^2}}{u^2} \cdot \frac{du}{3\cancel{x^2}}$ | $\frac{1}{3}\left(\frac{u^{-1}}{-1}\right) + C$

$u = 1+x^3$ | $= \frac{1}{3}\int \frac{1}{u^2}\,du$ | $\frac{-1}{3u} + C$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$ | $= \frac{1}{3}\int u^{-2}\,du$ | $= \boxed{\frac{-1}{3(1+x^3)} + C}$

23) $\int \frac{x}{\sqrt{1-x^2}}\,dx$

$\int \frac{x}{(1-x^2)^{1/2}}\,dx$

$u = 1-x^2$

$\frac{du}{dx} = -2x$

$dx = \frac{du}{-2x}$

$\int \frac{\cancel{x}}{u^{1/2}} \cdot \frac{du}{-2\cancel{x}}$

$\frac{-1}{2}\int \frac{1}{u^{1/2}}\,du$

$= \frac{-1}{2}\int u^{-1/2}\,du$

$= \frac{-1}{2} \cdot \frac{u^{1/2}}{1/2} + C$

$= \frac{-1}{2} \cdot 2u^{1/2} + C$

$= -1u^{1/2} + C$

$= \boxed{-(1-x^2)^{1/2} + C}$ or $-\sqrt{1-x^2} + C$

25) $\int \left(1+\frac{1}{t}\right)^3 \left(\frac{1}{t^2}\right) dt$

$u = 1 + \frac{1}{t}$

$u = 1 + t^{-1}$

$\frac{du}{dt} = -t^{-2}$

$\frac{du}{dt} = \frac{-1}{t^2}$

$dt = -t^2\,du$

$= \int (u)^3 \left(\frac{1}{\cancel{t^2}}\right) \cdot -\cancel{t^2}\,du$

$= -\int u^3\,du$

$= -\frac{u^4}{4} + C$

$= \boxed{\frac{-1}{4}\left(1+\frac{1}{t}\right)^4 + C}$

27) $\int \frac{1}{\sqrt{2x}}\,dx$

$= \int \frac{1}{(2x)^{1/2}}\,dx$

$u = 2x$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$= \int \frac{1}{u^{1/2}} \cdot \frac{du}{2}$

$= \frac{1}{2}\int u^{-1/2}\,du$

$= \frac{1}{2}\left(\frac{u^{1/2}}{1/2}\right) + C$

$= \frac{1}{2} \cdot 2 \cdot u^{1/2} + C$

$= u^{1/2} + C$

$= (2x)^{1/2} + C$

$= \boxed{\sqrt{2x} + C}$

29) $\int \frac{x^2+3x+7}{\sqrt{x}}\,dx$

$= \int \frac{x^2}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{7}{x^{1/2}}\,dx$

$= \int x^{3/2} + 3x^{1/2} + 7x^{-1/2}\,dx$

* No u-substitution needed!

$\frac{x^{5/2}}{5/2} + \frac{3x^{3/2}}{3/2} + \frac{7x^{1/2}}{1/2} + C$

$\frac{2}{5}x^{5/2} + 3\left(\frac{2}{3}\right)x^{3/2} + 7(2)x^{1/2} + C$

$= \boxed{\frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C}$

4.5a (Continued)

33) $\int (9-y)\sqrt{y}\,dy$ *Expand if possible to avoid u-substitution

$\int 9y^{1/2} - y^{3/2}\,dy$

$= \dfrac{9y^{3/2}}{3/2} - \dfrac{y^{5/2}}{5/2} + C$

$= 9\left(\frac{2}{3}\right)y^{3/2} - \frac{2}{5}y^{5/2} + C$

$= \boxed{6y^{3/2} - \frac{2}{5}y^{5/2} + C}$

47) $\int \frac{1}{\theta^2}\cos\left(\frac{1}{\theta}\right)d\theta$

$u = \frac{1}{\theta} = \theta^{-1}$

$\frac{du}{d\theta} = -1\theta^{-2}$

$\frac{du}{d\theta} = \frac{-1}{\theta^2}$

$d\theta = -\theta^2 du$

$= \int \frac{1}{\cancel{\theta^2}}\cos u \cdot -\cancel{\theta^2}\,du$

$= -\int \cos u\,du$

$= -\sin u + C$

$= \boxed{-\sin\left(\frac{1}{\theta}\right) + C}$

51) $\int \tan^4 x \sec^2 x\,dx$

$= \int (\tan x)^4 \sec^2 x\,dx$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

$= \int u^4 \cancel{\sec^2 x} \cdot \frac{du}{\cancel{\sec^2 x}}$

$= \int u^4\,du$

$= \frac{u^5}{5} + C$

$= \frac{1}{5}(\tan x)^5 + C$

$= \boxed{\frac{1}{5}\tan^5 x + C}$

53) $\int \frac{\csc^2 x}{\cot^3 x}\,dx$

$= \int \frac{\csc^2 x}{(\cot x)^3}\,dx$

$u = \cot x$

$\frac{du}{dx} = -\csc^2 x$

$dx = \frac{du}{-\csc^2 x}$

$= \int \frac{\cancel{\csc^2 x}}{u^3} \cdot \frac{du}{-\cancel{\csc^2 x}}$

$= -\int \frac{1}{u^3}\,du$

$= -\int u^{-3}\,du$

$= -\frac{u^{-2}}{-2} + C$

$= \frac{1}{2u^2} + C$

$= \boxed{\frac{1}{2(\cot x)^2} + C}$

or $\boxed{\frac{1}{2}\tan^2 x + C}$

55) $\int \cot^2 x\,dx$

* u-substitution does not work. Rewrite integrand using trig identities

$(1+\cot^2 x = \csc^2 x)$

$\downarrow$

$\cot^2 x = \csc^2 x - 1$

$= \int \csc^2 x - 1\,dx$

$= \boxed{-\cot x - x + C}$

Find equation for function with given derivative passing through point.

61) $f'(x) = 2x(4x^2-10)^2$ at point $(2,10)$

Steps:
1) Find indefinite integral to get $f(x)$
2) Use ordered pair to solve for C.

$\int 2x(4x^2-10)^2 dx$ $\quad u = 4x^2-10 \quad \frac{du}{dx} = 8x \quad dx = \frac{du}{8x}$

$\int 2x \cdot u^2 \cdot \frac{du}{8x}$

$\frac{1}{4}\int u^2 du$

$\frac{1}{4}\left(\frac{u^3}{3}\right) + C$

$\frac{1}{12}(4x^2-10)^3 + C$

*plug in (2,10) to find "C"

$f(x) = \frac{1}{12}(4x^2-10)^3 + C$

$10 = \frac{1}{12}(4(2)^2-10)^3 + C$

$10 = \frac{1}{12}(216) + C$

$10 = 18 + C \quad \underline{C = -8}$

$\boxed{f(x) = \frac{1}{12}(4x^2-10)^3 - 8}$

or $\frac{2}{3}(2x^2-5)^3 - 8$

Use change of variable u-substitution

63) $\int x\sqrt{x+2}\,dx$

$= \int x(x+2)^{1/2} dx$

$u = x+2$

$\frac{du}{dx} = 1$

$dx = du$

$\int x \cdot u^{1/2} du$

$u = x+2$

$x = u-2$

$\int (u-2)u^{1/2} du$

$= \int u^{3/2} - 2u^{1/2} du$

$\frac{u^{5/2}}{5/2} - \frac{2u^{3/2}}{3/2} + C$

$\frac{2}{5}u^{5/2} - 2\left(\frac{2}{3}\right)u^{3/2} + C$

$= \boxed{\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C}$

65) $\int x^2\sqrt{1-x}\,dx$

$= \int x^2(1-x)^{1/2} dx$

$u = 1-x$

$\frac{du}{dx} = -1$

$dx = -du$

$\int x^2 u^{1/2}(-du)$

$u = 1-x$

$x = 1-u$

$\int (1-u)^2 u^{1/2}(-du)$

$-\int (1-2u+u^2)u^{1/2} du$

$\int -u^{1/2} + 2u^{3/2} - u^{5/2} du$

$= \frac{-u^{3/2}}{3/2} + \frac{2u^{5/2}}{5/2} - \frac{u^{7/2}}{7/2} + C$

$= -\frac{2}{3}u^{3/2} + 2\left(\frac{2}{5}\right)u^{5/2} - \frac{2}{7}u^{7/2} + C$

$= \boxed{\frac{-2}{3}(1-x)^{3/2} + \frac{4}{5}(1-x)^{5/2} - \frac{2}{7}(1-x)^{7/2} + C}$

67) $\int \frac{x^2-1}{\sqrt{2x-1}}\,dx$

$\int \frac{x^2-1}{(2x-1)^{1/2}}dx$

$u=2x-1$

$\frac{du}{dx}=2$

$dx=\frac{du}{2}$

$\int \frac{x^2-1}{u^{1/2}}\left(\frac{du}{2}\right)$

$u=2x-1$

$\frac{u+1}{2}=x$

$=\int \frac{\left(\frac{u+1}{2}\right)^2-1}{u^{1/2}}\cdot\frac{du}{2}$

$=\int \frac{\frac{u^2+2u+1}{4}-\frac{4}{4}}{u^{1/2}}\left(\frac{du}{2}\right)$

$=\int \frac{u^2+2u-3}{4u^{1/2}}\left(\frac{du}{2}\right)$

$=\int \frac{u^2+2u-3}{8u^{1/2}}du$

$=\int \frac{u^2}{8u^{1/2}}+\frac{2u}{8u^{1/2}}-\frac{3}{8u^{1/2}}du$

$=\int \frac{1}{8}u^{3/2}+\frac{1}{4}u^{1/2}-\frac{3}{8}u^{-1/2}du$

$=\frac{1}{8}\left(\frac{u^{5/2}}{5/2}\right)+\frac{1}{4}\left(\frac{u^{3/2}}{3/2}\right)-\frac{3}{8}\left(\frac{u^{1/2}}{1/2}\right)$

$=\frac{2}{40}u^{5/2}+\frac{2}{12}u^{3/2}-\frac{6}{8}u^{1/2}+c$

$=\boxed{\frac{1}{20}(2x-1)^{5/2}+\frac{1}{6}(2x-1)^{3/2}-\frac{3}{4}(2x-1)^{1/2}+c}$

69) $\int \frac{-x}{(x+1)-\sqrt{x+1}}\,dx$

$u=x+1$ | $x=u-1$

$\frac{du}{dx}=1$

$dx=du$

$\int \frac{-x}{u-\sqrt{u}}du$

$\int \frac{-(u-1)}{u-\sqrt{u}}du$

$\int \frac{-u+1}{\sqrt{u}(\sqrt{u}-1)}du$

$\int \frac{1-u}{\sqrt{u}(\sqrt{u}-1)}du$

$\int \frac{\cancel{(1-\sqrt{u})}^{-1}(1+\sqrt{u})}{\sqrt{u}\,\cancel{(\sqrt{u}-1)}}du$

$\int \frac{-(1+\sqrt{u})}{\sqrt{u}}du$

$\int \frac{-1}{\sqrt{u}}-\frac{\sqrt{u}}{\sqrt{u}}du$

$=\int -u^{-1/2}-1\,du$

$=\frac{-u^{1/2}}{1/2}-u+c$

$=-2u^{1/2}-u+c$

$=-2(x+1)^{1/2}-(x+1)+c$

$=-2(x+1)^{1/2}-x-1+c$

$=\boxed{-2\sqrt{x+1}-x+c}$