102. $\alpha^2\int_0^1 f(x)\,dx = \alpha^2(1) = \alpha^2$

$$-2\alpha\int_0^1 f(x)x\,dx = -2\alpha(\alpha) = -2\alpha^2$$

$$\int_0^1 f(x)x^2\,dx = \alpha^2$$

Adding,

$$\int_0^1 \left[\alpha^2 f(x) - 2\alpha\, xf(x) + x^2 f(x)\right] dx = 0$$

$$\int_0^1 f(x)(\alpha - x)^2\,dx = 0.$$

Because $(\alpha - x)^2 \ge 0$, $f = 0$. So, there are no such functions.

Section 4.6 Numerical Integration

1. Exact: $\int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3} \approx 2.6667$

Trapezoidal: $\int_0^2 x^2\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{11}{4} = 2.7500$

Simpson's: $\int_0^2 x^2\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{8}{3} \approx 2.6667$

2. Exact: $\int_1^2 \left(\frac{x^2}{4} + 1\right) dx = \left[\frac{x^3}{12} + x\right]_1^2 = \frac{19}{12} \approx 1.5833$

Trapezoidal: $\int_1^2 \left(\frac{x^2}{4} + 1\right) dx \approx \frac{1}{8}\left[\left(\frac{1^2}{4} + 1\right) + 2\left(\frac{(5/4)^2}{4} + 1\right) + 2\left(\frac{(3/2)^2}{4} + 1\right) + 2\left(\frac{(7/4)^2}{4} + 1\right) + \left(\frac{2^2}{4} + 1\right)\right] = \frac{203}{128} \approx 1.5859$

Simpson's: $\int_0^1 \left(\frac{x^2}{4} + 1\right) dx \approx \frac{1}{12}\left[\left(\frac{1^2}{4} + 1\right) + 4\left(\frac{(5/4)^2}{4} + 1\right) + 2\left(\frac{(3/2)^2}{4} + 1\right) + 4\left(\frac{(7/4)^2}{4} + 1\right) + \left(\frac{2^2}{4} + 1\right)\right] = \frac{19}{12} \approx 1.5833$

3. Exact: $\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 4.0000$

Trapezoidal: $\int_0^2 x^3\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^3 + 2(1)^3 + 2\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{17}{4} = 4.2500$

Simpson's: $\int_0^2 x^3\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^3 + 2(1)^3 + 4\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{24}{6} = 4.0000$

4. Exact: $\int_2^3 \frac{2}{x^2}\,dx = \left[-\frac{2}{x}\right]_2^3 = -\frac{2}{3} + \frac{2}{2} = \frac{1}{3}$

Trapezoidal: $\int_2^3 \frac{2}{x^2}\,dx \approx \frac{1}{8}\left[\frac{2}{2^2} + 2\left(\frac{2}{(9/4)^2}\right) + 2\left(\frac{2}{(10/4)^2}\right) + 2\left(\frac{2}{(11/4)^2}\right) + \frac{2}{3^2}\right] \approx 0.3352$

Simpson's: $\int_2^3 \frac{2}{x^2}\,dx \approx \frac{1}{12}\left[\frac{2}{2^2} + 4\left(\frac{2}{(9/4)^2}\right) + 2\left(\frac{2}{(10/4)^2}\right) + 4\left(\frac{2}{(11/4)^2}\right) + \frac{2}{3^2}\right] \approx 0.3334$

5. Exact: $\int_1^3 x^3\,dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{81}{4} - \frac{1}{4} = 20$

Trapezoidal: $\int_1^3 x^3\,dx \approx \frac{1}{6}\left[1 + 2\left(\frac{4}{3}\right)^3 + 2\left(\frac{5}{3}\right)^3 + 2(2)^3 + 2\left(\frac{7}{3}\right)^3 + 2\left(\frac{8}{3}\right)^3 + 27\right] \approx 20.2222$

Simpson's: $\int_1^3 x^3\,dx \approx \frac{1}{9}\left[1 + 4\left(\frac{4}{3}\right)^3 + 2\left(\frac{5}{3}\right)^3 + 4(2)^3 + 2\left(\frac{7}{3}\right)^3 + 4\left(\frac{8}{3}\right)^3 + 27\right] = 20.0000$

6. Exact: $\int_0^8 \sqrt[3]{x}\,dx = \left[\frac{3}{4}x^{4/3}\right]_0^8 = 12.0000$

Trapezoidal: $\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{2}\left[0 + 2 + 2\sqrt[3]{2} + 2\sqrt[3]{3} + 2\sqrt[3]{4} + 2\sqrt[3]{5} + 2\sqrt[3]{6} + 2\sqrt[3]{7} + 2\right] \approx 11.7296$

Simpson's: $\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{3}\left[0 + 4 + 2\sqrt[3]{2} + 4\sqrt[3]{3} + 2\sqrt[3]{4} + 4\sqrt[3]{5} + 2\sqrt[3]{6} + 4\sqrt[3]{7} + 2\right] \approx 11.8632$

7. Exact: $\int_4^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$

Trapezoidal: $\int_4^9 \sqrt{x}\,dx \approx \frac{5}{16}\left[2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3\right] \approx 12.6640$

Simpson's: $\int_4^9 \sqrt{x}\,dx \approx \frac{5}{24}\left[2 + 4\sqrt{\frac{37}{8}} + \sqrt{21} + 4\sqrt{\frac{47}{8}} + \sqrt{26} + 4\sqrt{\frac{57}{8}} + \sqrt{31} + 4\sqrt{\frac{67}{8}} + 3\right] \approx 12.6667$

8. Exact: $\int_1^4 \left(4 - x^2\right)dx = \left[4x - \frac{x^3}{3}\right]_1^4 = -\frac{16}{3} - \frac{11}{3} = -9$

Trapezoidal: $\int_1^4 \left(4 - x^2\right)dx \approx \frac{1}{4}\left\{3 + 2\left[4 - \left(\frac{3}{2}\right)^2\right] + 2(0) + 2\left[4 - \left(\frac{5}{2}\right)^2\right] + 2(-5) + 2\left[4 - \left(\frac{7}{2}\right)^2\right] - 12\right\} \approx -9.1250$

Simpson's: $\int_1^4 \left(4 - x^2\right)dx \approx \frac{1}{6}\left[3 + 4\left(4 - \frac{9}{4}\right) + 0 + 4\left(4 - \frac{25}{4}\right) - 10 + 4\left(4 - \frac{49}{4}\right) - 12\right] = -9$

9. Exact: $\int_0^1 \frac{2}{(x+2)^2}\,dx = \left[\frac{-2}{(x+2)}\right]_0^1 = \frac{-2}{3} + \frac{2}{2} = \frac{1}{3}$

Trapezoidal: $\int_0^1 \frac{2}{(x+2)^2}\,dx \approx \frac{1}{8}\left[\frac{1}{2} + 2\left(\frac{2}{((1/4)+2)^2}\right) + 2\left(\frac{2}{((1/2)+2)^2}\right) + 2\left(\frac{2}{((3/4)+2)^2}\right) + \frac{2}{9}\right]$

$= \frac{1}{8}\left[\frac{1}{2} + 2\left(\frac{32}{81}\right) + 2\left(\frac{8}{25}\right) + 2\left(\frac{32}{121}\right) + \frac{2}{9}\right] \approx 0.3352$

Simpson's: $\int_0^1 \frac{2}{(x+2)^2}\,dx \approx \frac{1}{12}\left[\frac{1}{2} + 4\left(\frac{2}{((1/4)+2)^2}\right) + 2\left(\frac{2}{((1/2)+2)^2}\right) + 4\left(\frac{2}{((3/4)+2)^2}\right) + \frac{2}{9}\right]$

$= \frac{1}{12}\left[\frac{1}{2} + 4\left(\frac{32}{81}\right) + 2\left(\frac{8}{25}\right) + 4\left(\frac{32}{121}\right) + \frac{2}{9}\right] \approx 0.3334$

10. Exact: $\int_0^2 x\sqrt{x^2+1}\,dx = \frac{1}{3}\left[\left(x^2+1\right)^{3/2}\right]_0^2 = \frac{1}{3}\left(5^{3/2}-1\right) \approx 3.393$

Trapezoidal: $\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2+1} + 2(1)\sqrt{1^2+1} + 2\left(\frac{3}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2+1} + 2\sqrt{2^2+1}\right] \approx 3.457$

Simpson's: $\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2+1} + 2(1)\sqrt{1^2+1} + 4\left(\frac{3}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2+1} + 2\sqrt{2^2+1}\right] \approx 3.392$

11. Trapezoidal: $\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{4}\left[1 + 2\sqrt{1+\left(\frac{1}{8}\right)} + 2\sqrt{2} + 2\sqrt{1+\left(\frac{27}{8}\right)} + 3\right] \approx 3.283$

Simpson's: $\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{6}\left[1 + 4\sqrt{1+\left(\frac{1}{8}\right)} + 2\sqrt{2} + 4\sqrt{1+\left(\frac{27}{8}\right)} + 3\right] \approx 3.240$

Graphing utility: 3.241

12. Trapezoidal: $\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{4}\left[1 + 2\left(\frac{1}{\sqrt{1+(1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1+1^3}}\right) + 2\left(\frac{1}{\sqrt{1+(3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.397$

Simpson's: $\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{6}\left[1 + 4\left(\frac{1}{\sqrt{1+(1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1+1^3}}\right) + 4\left(\frac{1}{\sqrt{1+(3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.405$

Graphing utility: 1.402

13. $\int_0^1 \sqrt{x}\sqrt{1-x}\,dx = \int_0^1 \sqrt{x(1-x)}\,dx$

Trapezoidal: $\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{8}\left[0 + 2\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.342$

Simpson's: $\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{12}\left[0 + 4\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 4\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.372$

Graphing utility: 0.393

14. Trapezoidal: $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{16}\left[\sqrt{\frac{\pi}{2}}(1) + 2\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 2\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.430$

Simpson's: $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{24}\left[\sqrt{\frac{\pi}{2}} + 4\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 4\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.458$

Graphing utility: 1.458

15. Trapezoidal: $\int_0^{\sqrt{\pi/2}} \sin\left(x^2\right)dx \approx \frac{\sqrt{\pi/2}}{8}\left[\sin 0 + 2\sin\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\sin\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2\sin\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \sin\left(\sqrt{\frac{\pi}{2}}\right)^2\right] \approx 0.550$

Simpson's: $\int_0^{\sqrt{\pi/2}} \sin\left(x^2\right)dx \approx \frac{\sqrt{\pi/2}}{12}\left[\sin 0 + 4\sin\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\sin\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 4\sin\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \sin\left(\sqrt{\frac{\pi}{2}}\right)^2\right] \approx 0.548$

Graphing utility: 0.549

16. Trapezoidal: $\int_0^{\sqrt{\pi/4}} \tan(x^2)\,dx \approx \frac{\sqrt{\pi/4}}{8}\left[\tan 0 + 2\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 2\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right] \approx 0.271$

Simpson's: $\int_0^{\sqrt{\pi/4}} \tan(x^2)\,dx \approx \frac{\sqrt{\pi/4}}{12}\left[\tan 0 + 4\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 4\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right] \approx 0.257$

Graphing utility: 0.256

17. Trapezoidal: $\int_3^{3.1} \cos x^2\,dx \approx \frac{0.1}{8}\left[\cos(3)^2 + 2\cos(3.025)^2 + 2\cos(3.05)^2 + 2\cos(3.075)^2 + \cos(3.1)^2\right] \approx -0.098$

Simpson's: $\int_3^{3.1} \cos x^2\,dx \approx \frac{0.1}{12}\left[\cos(3)^2 + 4\cos(3.025)^2 + 2\cos(3.05)^2 + 4\cos(3.075)^2 + \cos(3.1)^2\right] \approx -0.098$

Graphing utility: −0.098

18. Trapezoidal: $\int_0^{\pi/2} \sqrt{1+\sin^2 x}\,dx \approx \frac{\pi}{16}\left[1 + 2\sqrt{1+\sin^2\left(\frac{\pi}{8}\right)} + 2\sqrt{1+\sin^2\left(\frac{\pi}{4}\right)} + 2\sqrt{1+\sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{2}\right] \approx 1.910$

Simpson's: $\int_0^{\pi/2} \sqrt{1+\sin^2 x}\,dx \approx \frac{\pi}{24}\left[1 + 4\sqrt{1+\sin^2\left(\frac{\pi}{8}\right)} + 2\sqrt{1+\sin^2\left(\frac{\pi}{4}\right)} + 4\sqrt{1+\sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{2}\right] \approx 1.910$

Graphing utility: 1.910

19. Trapezoidal: $\int_0^{\pi/4} x\tan x\,dx \approx \frac{\pi}{32}\left[0 + 2\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 2\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.194$

Simpson's: $\int_0^{\pi/4} x\tan x\,dx \approx \frac{\pi}{48}\left[0 + 4\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 4\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.186$

Graphing utility: 0.186

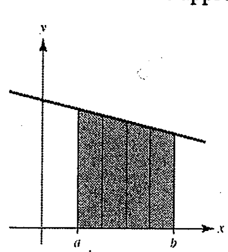
20. Trapezoidal: $\int_0^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{8}\left[1 + \frac{2\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{2\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.836$

Simpson's: $\int_0^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{12}\left[1 + \frac{4\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{4\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.852$

Graphing utility: 1.852

21. Trapezoidal: Linear polynomials
Simpson's: Quadratic polynomials

22. For a linear function, the Trapezoidal Rule is exact. The error formula says that $E \le \frac{(b-a)^3}{12n^2}\left[\max|f''(x)|\right]$ and $f''(x) = 0$ for a linear function. Geometrically, a linear function is approximated exactly by trapezoids:



23.
$f(x) = 2x^3$
$f'(x) = 6x^2$
$f''(x) = 12x$
$f'''(x) = 12$
$f^{(4)}(x) = 0$

(a) Trapezoidal: Error $\le \frac{(3-1)^3}{12(4^2)}(36) = 1.5$ because $|f''(x)|$ is maximum in $[1, 3]$ when $x = 3$.

(b) Simpson's: Error $\le \frac{(3-1)^5}{180(4^4)}(0) = 0$ because $f^{(4)}(x) = 0$.