5.03 Notes: Introduction to Matrix Inverses

What's created when you multiply inverses together?

$2\left(\frac{1}{2}\right) = 1$ $\quad 3\left(\frac{1}{3}\right) = 1$ $\quad 25\left(\frac{1}{25}\right) = 1$

Multiply matrix inverses together and you get….. $[A][B] = I$ and $[B][A] = I$

Identity Matrix: a square matrix with diagonal elements = 1 and all other elements = 0

$I_1 = [1]$ $\quad I_2 = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$ $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Multiplying any matrix by the identity matrix is like multiplying a number by 1

Ex: $3 \times 1 = 3$

Ex. $10 \times 1 = 10$

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 & -2 \\ 1 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 6(1)+0(1) & -2(1)+0(-1) \\ 6(0)+1(1) & 0(-2)+1(-1) \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 1 & -1 \end{bmatrix}$$

Why?

* Every number not part of the original matrix gets zeroed out, leaving us with the original matrix unchanged.

Find the product of the given matrices to determine if they are inverses.

$A = \begin{bmatrix} -7 & 2 \\ -8 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 4 & -3.5 \end{bmatrix}$

$$\begin{bmatrix} -7 & 2 \\ -8 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 4 & -3.5 \end{bmatrix} = \begin{bmatrix} -7+8 & 7+2(-3.5) \\ -8(1)+2(4) & -8(-1)+2(-3.5) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

yes, matrix A and B are inverses.

$C = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 3 \\ -1 & -2 & -8 \end{bmatrix}$ and $D = \begin{bmatrix} -6 & -4 & 3 \\ 11 & 6 & -5 \\ -2 & -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 3 \\ -1 & -2 & -8 \end{bmatrix}\begin{bmatrix} -6 & -4 & 3 \\ 11 & 6 & -5 \\ -2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -6+11-4 & -4+6-2 & 3-5+2 \\ 6+0-6 & 4+0-3 & -3+0+3 \\ 6-22+16 & 4-12+8 & -3+10-8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

C and D are not inverses of each other

important component for finding inverse of matrix

Determinant of a 2x2 Matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

For $E = \begin{bmatrix} 6 & 1 \\ -2 & 5 \end{bmatrix}$, find det($E$)

$\det[E] = |E| = 6(5) - (-2)(1)$

$= 30 + 2 = \boxed{32}$

For $F = \begin{bmatrix} 3 & 9 \\ -2 & -6 \end{bmatrix}$, find $|F|$

$|F| = 3(-6) - (-2)(9) = -18 - (-18)$

$= -18 + 18 = \boxed{0}$ *matrix F does not have inverse

5.03 Practice

Determine whether A and B are inverse matrices.

1. $A = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix}$

$$\begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix} = \begin{bmatrix} 12(3)+-7(5) & 12(7)-7(12) \\ -5(3)+3(5) & -5(7)+3(12) \end{bmatrix}$$

Identity Matrix → $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ yes since $[A][B]=I$

2. $A = \begin{bmatrix} 2 & 3 & -4 \\ 3 & 6 & -5 \\ -2 & -8 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -34 & 29 & 9 \\ 7 & -6 & -2 \\ -12 & 10 & 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & 6 & -5 \\ -2 & -8 & 1 \end{bmatrix}\begin{bmatrix} -34 & 29 & 9 \\ 7 & -6 & -2 \\ -12 & 10 & 3 \end{bmatrix} = \begin{bmatrix} -68+21+48 & 58-18-40 & 18-6-12 \\ -102+42+60 & 87-36-50 & 27-12-15 \\ 68-56-12 & -58+48+10 & -18+16+3 \end{bmatrix}$$

yes, since $[A][B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Find the determinant of each matrix.

3. $\begin{bmatrix} 6 & -5 \\ 3 & -2 \end{bmatrix}$

$\det(A) = 6(-2)-(-5)(3)$
$= -12+15$
$= \boxed{3}$

4. $\begin{bmatrix} -2 & 7 \\ 1 & 8 \end{bmatrix}$

$\det(A) = -2(8)-7(1)$
$= -16-7$
$= \boxed{-23}$

5. $\begin{bmatrix} -4 & -7 \\ 6 & 9 \end{bmatrix}$

$\det(A) = -4(9)-(-7)(6)$
$= -36+42$
$= \boxed{6}$

6. $\begin{bmatrix} 12 & -9 \\ -4 & 3 \end{bmatrix}$

$\det(A) = 12(3)-(-9)(-4)$
$= 36-36 = \boxed{0}$

5.04 Matrix Inverses Notes

Things to Know:

Matrix Multiplicative Identity: $X \cdot I = X$ and $I \cdot X = X$

Determinants: Value used to find the inverse

Inverses: (notation: A^{-1}) $A \cdot A^{-1} = I$ $A^{-1} \cdot A = I$

* the "-b" and "-c" means those values will be the opposite signs of the original starting matrix. (not necessarily negative values)

Finding the inverse of a matrix:

-A matrix is considered non-singular if it has an inverse and singular if it does not have an inverse

-A matrix is singular and has no inverse if the determinant = 0

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ 8 & -7 \end{bmatrix} =$

Ex: $A = \begin{bmatrix} -7 & 2 \\ -8 & 2 \end{bmatrix}$ $det = -14-(-16)$
$det = 2$

$\boxed{A^{-1} = \begin{bmatrix} 1 & -1 \\ 4 & -3.5 \end{bmatrix}}$

Find the determinant. Then find the inverse of the matrix, if it exists.

1. $A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$ $det = 8-(-12) = 20$

$A^{-1} = \frac{1}{20}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1/5 & 3/20 \\ -1/5 & \frac{1}{10} \end{bmatrix}$

2. $B = \begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}$ $det\, B = 36-36 = 0$

No inverse for matrix B b/c matrix B is singular.

3. $C = \begin{bmatrix} -4 & 6 \\ 8 & -12 \end{bmatrix}$ $det\, C = 48-48 = 0$

no inverse for matrix C

4. $D = \begin{bmatrix} 2 & -3 \\ 2 & 2 \end{bmatrix}$ $det(D) = 4-(-6) = 10$

$D^{-1} = \frac{1}{10}\begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1/5 & 3/10 \\ -1/5 & 1/5 \end{bmatrix}$

Given $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$ and $AB = \begin{bmatrix} 30 & -20 \\ 21 & -14 \end{bmatrix}$ find B. (Hint: remember that the product of inverses is the identity matrix)

$[A] \cdot [B] = AB$
$[B] = [A^{-1}] \cdot AB$

$det(A) = 2(4) - 6(1)$
$= 8-6 = 2$

$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -\frac{1}{2} & 1 \end{bmatrix}$

$B = \begin{bmatrix} 2 & -3 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} 30 & -20 \\ 21 & -14 \end{bmatrix}$

$\begin{bmatrix} 2 & -3 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} 2(30)-3(21) & 2(-20)+3(14) \\ -1/2(30)+21 & -1/2(-20)+1(-14) \end{bmatrix} = \boxed{\begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix}}$

5.04 Practice $A^{-1} = \frac{1}{det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

In #1−4, use the determinants you found yesterday in #3−6 to find the inverse of the given matrix, if it exists.

1. $\begin{bmatrix} 6 & -5 \\ 3 & -2 \end{bmatrix}$ $det(A)=3$

$A^{-1} = \frac{1}{3}\begin{bmatrix} -2 & 5 \\ -3 & 6 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -2/3 & 5/3 \\ -1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} -2 & 7 \\ 1 & 8 \end{bmatrix}$ $det(A) = -23$

$A^{-1} = \frac{-1}{23}\begin{bmatrix} 8 & -7 \\ -1 & -2 \end{bmatrix}$

$\boxed{A^{-1} = \begin{bmatrix} -8/23 & 7/23 \\ 1/23 & 2/23 \end{bmatrix}}$

3. $\begin{bmatrix} -4 & -7 \\ 6 & 9 \end{bmatrix}$ $det(A)=6$

$A^{-1} = \frac{1}{6}\begin{bmatrix} 9 & 7 \\ -6 & -4 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 9/6 & 7/6 \\ -1 & -4/6 \end{bmatrix}$ or $\boxed{\begin{bmatrix} 3/2 & 7/6 \\ -1 & -2/3 \end{bmatrix}}$

4. $\begin{bmatrix} 12 & -9 \\ -4 & 3 \end{bmatrix}$ $det(A)=0$

$\boxed{\text{no inverse exists for matrix}}$

5. Given A and AB, find B. $A = \begin{bmatrix} 8 & -4 \\ 3 & 6 \end{bmatrix}$, and $AB = \begin{bmatrix} 36 & 48 \\ -24 & 48 \end{bmatrix}$

$[A]\cdot[B] = AB$

$[B] = [A]^{-1}\cdot AB$

$det(A) = 8(6)-(-4)(3)$
$= 60$

$A^{-1} = \boxed{\frac{1}{60}\begin{bmatrix} 6 & 4 \\ -3 & 8 \end{bmatrix}} = \begin{bmatrix} 1/10 & 1/15 \\ -1/20 & 2/15 \end{bmatrix}$

$[B] = \frac{1}{60}\begin{bmatrix} 6 & 4 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} 36 & 48 \\ -24 & 48 \end{bmatrix}$

$\begin{bmatrix} 36 & 48 \\ -24 & 48 \end{bmatrix}$

$\begin{bmatrix} 6 & 4 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} 6(36)+4(-24) & 6(48)+4(48) \\ -3(36)+8(-24) & -3(48)+8(48) \end{bmatrix} \rightarrow \begin{bmatrix} 120 & 480 \\ -300 & 240 \end{bmatrix}$

$[B] = \frac{1}{60}\begin{bmatrix} 120 & 480 \\ -300 & 240 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 8 \\ -5 & 4 \end{bmatrix}}$