CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

Section 5.1 The Natural Logarithmic Function: Differentiation

1. (a) $\ln 45 \approx 3.8067$

   (b) $\int_1^{45} \frac{1}{t}\,dt \approx 3.8067$

2. (a) $\ln 8.3 \approx 2.1163$

   (b) $\int_1^{8.3} \frac{1}{t}\,dt \approx 2.1163$

3. (a) $\ln 0.8 \approx -0.2231$

   (b) $\int_1^{0.8} \frac{1}{t}\,dt \approx -0.2231$

4. (a) $\ln 0.6 \approx -0.5108$

   (b) $\int_1^{0.6} \frac{1}{t}\,dt \approx -0.5108$

5. $f(x) = \ln x + 1$

   Vertical shift 1 unit upward

   Matches (b)

6. $f(x) = -\ln x$

   Reflection in the x-axis

   Matches (d)

7. $f(x) = \ln(x - 1)$

   Horizontal shift 1 unit to the right

   Matches (a)

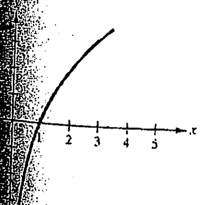
8. $f(x) = -\ln(-x)$

   Reflection in the y-axis and the x-axis

   Matches (c)

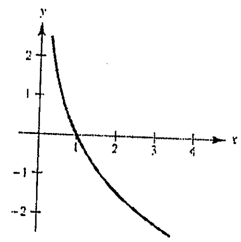
9. $f(x) = 3\ln x$

   Domain: $x > 0$



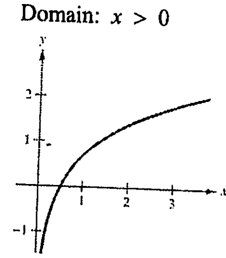
10. $f(x) = -2\ln x$

    Domain: $x > 0$



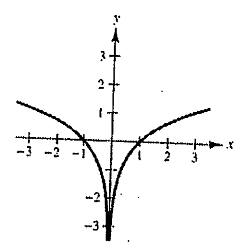
11. $f(x) = \ln 2x$

    Domain: $x > 0$



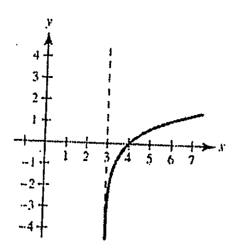
12. $f(x) = \ln|x|$

    Domain: $x \neq 0$



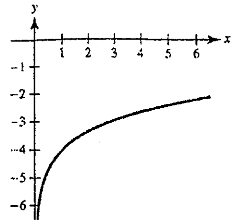
13. $f(x) = \ln(x - 3)$

    Domain: $x > 3$



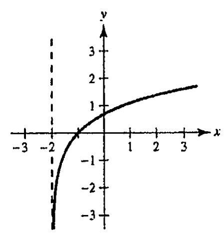
14. $f(x) = \ln x - 4$

Domain: $x > 0$



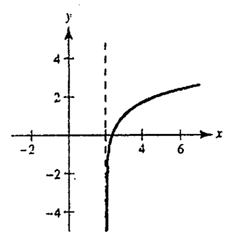
15. $h(x) = \ln(x + 2)$

Domain: $x > -2$



16. $f(x) = \ln(x - 2) + 1$

Domain: $x > 2$



17. (a) $\ln 6 = \ln 2 + \ln 3 \approx 1.7917$

(b) $\ln \frac{2}{3} = \ln 2 - \ln 3 \approx -0.4055$

(c) $\ln 81 = \ln 3^4 = 4 \ln 3 \approx 4.3944$

(d) $\ln \sqrt{3} = \ln 3^{1/2} = \frac{1}{2} \ln 3 \approx 0.5493$

18. (a) $\ln 0.25 = \ln \frac{1}{4} = \ln 1 - 2 \ln 2 \approx -1.3862$

(b) $\ln 24 = 3 \ln 2 + \ln 3 \approx 3.1779$

(c) $\ln \sqrt[3]{12} = \frac{1}{3}(2 \ln 2 + \ln 3) \approx 0.8283$

(d) $\ln \frac{1}{72} = \ln 1 - (3 \ln 2 + 2 \ln 3) \approx -4.2765$

19. $\ln \frac{x}{4} = \ln x - \ln 4$

20. $\ln \sqrt{x^5} = \ln x^{5/2} = \frac{5}{2} \ln x$

21. $\ln \frac{xy}{z} = \ln x + \ln y - \ln z$

22. $\ln(xyz) = \ln x + \ln y + \ln z$

23. $\ln\left(x\sqrt{x^2 + 5}\right) = \ln x + \ln\left(x^2 + 5\right)^{1/2}$

$= \ln x + \frac{1}{2} \ln\left(x^2 + 5\right)$

24. $\ln \sqrt{a - 1} = \ln(a - 1)^{1/2} = \left(\frac{1}{2}\right) \ln(a - 1)$

25. $\ln\sqrt{\frac{x - 1}{x}} = \ln\left(\frac{x - 1}{x}\right)^{1/2} = \frac{1}{2} \ln\left(\frac{x - 1}{x}\right)$

$= \frac{1}{2}\left[\ln(x - 1) - \ln x\right]$

$= \frac{1}{2} \ln(x - 1) - \frac{1}{2} \ln x$

26. $\ln 3e^2 = \ln 3 + 2 \ln e = 2 + \ln 3$

27. $\ln z(z - 1)^2 = \ln z + \ln(z - 1)^2$

$= \ln z + 2 \ln(z - 1)$

28. $\ln \frac{1}{e} = \ln 1 - \ln e = -1$

29. $\ln(x - 2) - \ln(x + 2) = \ln \frac{x - 2}{x + 2}$

30. $3 \ln x + 2 \ln y - 4 \ln z = \ln x^3 + \ln y^2 - \ln z^4$

$= \ln \frac{x^3 y^2}{z^4}$

31. $\frac{1}{3}\left[2\ln(x+3) + \ln x - \ln(x^2-1)\right] = \frac{1}{3}\ln\frac{x(x+3)^2}{x^2-1}$

$$= \ln\sqrt[3]{\frac{x(x+3)^2}{x^2-1}}$$

32. $2\left[\ln x - \ln(x+1) - \ln(x-1)\right] = 2\ln\frac{x}{(x+1)(x-1)}$

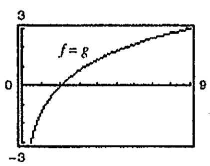
$$= \ln\left(\frac{x}{x^2-1}\right)^2$$

33. $2\ln 3 - \frac{1}{2}\ln(x^2+1) = \ln 9 - \ln\sqrt{x^2+1} = \ln\frac{9}{\sqrt{x^2+1}}$

34. $\frac{3}{2}\left[\ln(x^2+1) - \ln(x+1) - \ln(x-1)\right] = \frac{3}{2}\ln\frac{x^2+1}{(x+1)(x-1)}$

$$= \ln\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^3}$$

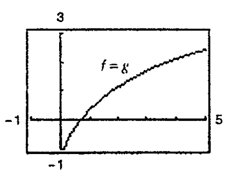
35. (a)



(b) $f(x) = \ln\frac{x^2}{4} = \ln x^2 - \ln 4 = 2\ln x - \ln 4 = g(x)$

because $x > 0$.

36. (a)



(b) $f(x) = \ln\sqrt{x(x^2+1)} = \frac{1}{2}\ln\left[x(x^2+1)\right]$

$$= \frac{1}{2}\left[\ln x + \ln(x^2+1)\right] = g(x)$$

37. $\lim_{x\to 3^+}\ln(x-3) = -\infty$

38. $\lim_{x\to 6^-}\ln(6-x) = -\infty$

39. $\lim_{x\to 2^-}\ln\left[x^2(3-x)\right] = \ln 4 \approx 1.3863$

40. $\lim_{x\to 5^+}\ln\frac{x}{\sqrt{x-4}} = \ln 5 \approx 1.6094$

41. $f(x) = \ln(3x)$

$f'(x) = \frac{1}{3x}(3) = \frac{1}{x}$

42. $f(x) = \ln(x-1)$

$f'(x) = \frac{1}{x-1}$

43. $g(x) = \ln x^2 = 2\ln x$

$g'(x) = \frac{2}{x}$

44. $h(x) = \ln(2x^2+1)$

$h'(x) = \frac{1}{2x^2+1}(4x) = \frac{4x}{2x^2+1}$

45. $y = (\ln x)^4$

$\frac{dy}{dx} = 4(\ln x)^3\left(\frac{1}{x}\right) = \frac{4(\ln x)^3}{x}$

46. $y = x^2\ln x$

$y' = x^2\left(\frac{1}{x}\right) + 2x\ln x = x + 2x\ln x = x(1 + 2\ln x)$

47. $y = \ln(t+1)^2 = 2\ln(t+1)$

$y' = 2\frac{1}{t+1} = \frac{2}{t+1}$

48. $y = \ln\sqrt{x^2-4} = \frac{1}{2}\ln(x^2-4)$

$\frac{dy}{dx} = \frac{1}{2}\left(\frac{2x}{x^2-4}\right) = \frac{x}{x^2-4}$

49. $y = \ln\left[x\sqrt{x^2 - 1}\right] = \ln x + \frac{1}{2}\ln(x^2 - 1)$

$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\left(\frac{2x}{x^2 - 1}\right) = \frac{2x^2 - 1}{x(x^2 - 1)}$

50. $y = \ln\left[t(t^2 + 3)^3\right] = \ln t + 3\ln(t^2 + 3)$

$y' = \frac{1}{t} + \frac{3}{t^2 + 3}(2t) = \frac{1}{t} + \frac{6t}{t^2 + 3}$

51. $f(x) = \ln\frac{x}{x^2 + 1} = \ln x - \ln(x^2 + 1)$

$f'(x) = \frac{1}{x} - \frac{2x}{x^2 + 1} = \frac{1 - x^2}{x(x^2 + 1)}$

52. $f(x) = \ln\left(\frac{2x}{x + 3}\right) = \ln 2x - \ln(x + 3)$

$f'(x) = \frac{1}{x} - \frac{1}{x + 3} = \frac{3}{x(x + 3)}$

53. $g(t) = \frac{\ln t}{t^2}$

$g'(t) = \frac{t^2(1/t) - 2t\ln t}{t^4} = \frac{1 - 2\ln t}{t^3}$

54. $h(t) = \frac{\ln t}{t}$

$h'(t) = \frac{t(1/t) - \ln t}{t^2} = \frac{1 - \ln t}{t^2}$

55. $y = \ln(\ln x^2)$

$\frac{dy}{dx} = \frac{1}{\ln x^2}\frac{d}{dx}(\ln x^2) = \frac{(2x/x^2)}{\ln x^2} = \frac{2}{x\ln x^2} = \frac{1}{x\ln x}$

56. $y = \ln(\ln x)$

$\frac{dy}{dx} = \frac{1/x}{\ln x} = \frac{1}{x\ln x}$

57. $y = \ln\sqrt{\frac{x + 1}{x - 1}} = \frac{1}{2}\left[\ln(x + 1) - \ln(x - 1)\right]$

$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x + 1} - \frac{1}{x - 1}\right] = \frac{1}{1 - x^2}$

58. $y = \ln\sqrt[3]{\frac{x - 1}{x + 1}} = \frac{1}{3}\left[\ln(x - 1) - \ln(x + 1)\right]$

$y' = \frac{1}{3}\left[\frac{1}{x - 1} - \frac{1}{x + 1}\right] = \frac{1}{3}\frac{2}{x^2 - 1} = \frac{2}{3(x^2 - 1)}$

59. $f(x) = \ln\frac{\sqrt{4 + x^2}}{x} = \frac{1}{2}\ln(4 + x^2) - \ln x$

$f'(x) = \frac{x}{4 + x^2} - \frac{1}{x} = \frac{-4}{x(x^2 + 4)}$

60. $f(x) = \ln\left(x + \sqrt{4 + x^2}\right)$

$f'(x) = \frac{1}{x + \sqrt{4 + x^2}}\left(1 + \frac{x}{\sqrt{4 + x^2}}\right) = \frac{1}{\sqrt{4 + x^2}}$

61. $y = \ln|\sin x|$

$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$

62. $y = \ln|\csc x|$

$y' = \frac{-\csc x \cdot \cot x}{\csc x} = -\cot x$

63. $y = \ln\left|\frac{\cos x}{\cos x - 1}\right|$

$= \ln|\cos x| - \ln|\cos x - 1|$

$\frac{dy}{dx} = \frac{-\sin x}{\cos x} - \frac{-\sin x}{\cos x - 1}$

$= -\tan x + \frac{\sin x}{\cos x - 1}$

64. $y = \ln|\sec x + \tan x|$

$\frac{dy}{dx} = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$

$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$

65. (a) $y = \ln x^4 = 4\ln x, (1, 0)$

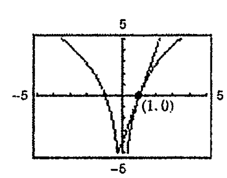
$\frac{dy}{dx} = \frac{4}{x}$

When $x = 1, \frac{dy}{dx} = 4.$

Tangent line: $y - 0 = 4(x - 1)$

$y = 4x - 4$

(b)



66. (a) $y = \ln x^{3/2} = \frac{3}{2}\ln x, \ (1, 0)$

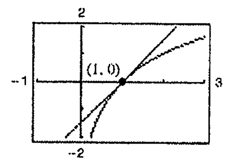
$$\frac{dy}{dx} = \frac{3}{2x}$$

When $x = 1, \frac{dy}{dx} = \frac{3}{2}$.

Tangent line: $y - 0 = \frac{3}{2}(x - 1)$

$$y = \frac{3}{2}x - \frac{3}{2}$$

(b)



67. (a) $y = 3x^2 - \ln x, \quad (1, 3)$

$$\frac{dy}{dx} = 6x - \frac{1}{x}$$

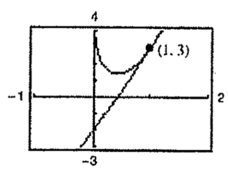
When $x = 1, \frac{dy}{dx} = 5$.

Tangent line: $y - 3 = 5(x - 1)$

$$y = 5x - 2$$

$$0 = 5x - y - 2$$

(b)



68. (a) $y = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right), \quad (0, 4)$

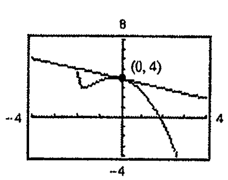
$$\frac{dy}{dx} = -2x - \frac{1}{(1/2)x + 1}\left(\frac{1}{2}\right) = -2x - \frac{1}{x + 2}$$

When $x = 0, \frac{dy}{dx} = -\frac{1}{2}$.

Tangent line: $y - 4 = -\frac{1}{2}(x - 0)$

$$y = -\frac{1}{2}x + 4$$

(b)



69. (a) $f(x) = \ln\sqrt{1 + \sin^2 x}$

$$= \frac{1}{2}\ln\left(1 + \sin^2 x\right), \quad \left(\frac{\pi}{4}, \ln\sqrt{\frac{3}{2}}\right)$$

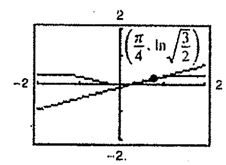
$$f'(x) = \frac{2\sin x\cos x}{2\left(1 + \sin^2 x\right)} = \frac{\sin x \cos x}{1 + \sin^2 x}$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\left(\sqrt{2}/2\right)\left(\sqrt{2}/2\right)}{(3/2)} = \frac{1}{3}$$

Tangent line: $y - \ln\sqrt{\frac{3}{2}} = \frac{1}{3}\left(x - \frac{\pi}{4}\right)$

$$y = \frac{1}{3}x + \frac{1}{2}\ln\left(\frac{3}{2}\right) - \frac{\pi}{12}$$

(b)



70. (a) $f(x) = \sin(2x)\ln\left(x^2\right) = 2\sin(2x)\ln x, \quad (1, 0)$

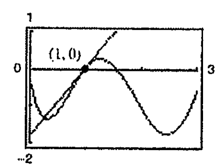
$$f'(x) = 4\cos(2x)\ln x + \frac{2\sin(2x)}{x}$$

$$f'(1) = 2\sin(2)$$

Tangent line: $y - 0 = 2\sin(2)(x - 1)$

$$y = 2\sin(2)x - 2\sin(2)$$

(b)



71. (a) $f(x) = x^3 \ln x, \quad (1, 0)$

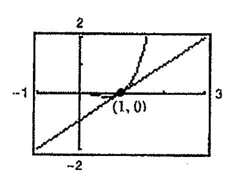
$$f'(x) = 3x^2\ln x + x^2$$

$$f'(1) = 1$$

Tangent line: $y - 0 = 1(x - 1)$

$$y = x - 1$$

(b)



72. (a) $f(x) = \frac{1}{2}x\ln(x^2), \quad (-1, 0)$

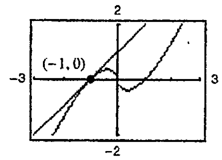
$$f'(x) = \frac{1}{2}\ln(x^2) + \frac{1}{2}x\left(\frac{2x}{x^2}\right) = \frac{1}{2}\ln(x^2) + 1$$

$$f'(-1) = 1$$

Tangent line: $y - 0 = 1(x + 1)$

$$y = x + 1$$

(b)



73. $x^2 - 3\ln y + y^2 = 10$

$$2x - \frac{3}{y}\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx}\left(\frac{3}{y} - 2y\right)$$

$$\frac{dy}{dx} = \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}$$

74. $\ln(xy) + 5x = 30$

$$\ln x + \ln y + 5x = 30$$

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + 5 = 0$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x} - 5$$

$$\frac{dy}{dx} = -\frac{y}{x} - 5y = -\left(\frac{y + 5xy}{x}\right)$$

75. $4x^3 + \ln y^2 + 2y = 2x$

$$12x^2 + \frac{2}{y}y' + 2y' = 2$$

$$\left(\frac{2}{y} + 2\right)y' = 2 - 12x^2$$

$$y' = \frac{2 - 12x^2}{2/y + 2}$$

$$y' = \frac{y - 6yx^2}{1 + y} = \frac{y(1 - 6x^2)}{1 + y}$$

76. $4xy + \ln x^2y = 7$

$$4xy + 2\ln x + \ln y = 7$$

$$4xy' + 4y + \frac{2}{x} + \frac{1}{y}y' = 0$$

$$\left(4x + \frac{1}{y}\right)y' = -4y - \frac{2}{x}$$

$$y' = \frac{-4y - \frac{2}{x}}{4x + \frac{1}{y}}$$

$$y' = \frac{-4xy^2 - 2y}{4x^2y + x}$$

77. $y = 2(\ln x) + 3$

$$y' = \frac{2}{x}$$

$$y'' = -\frac{2}{x^2}$$

$$xy'' + y' = x\left(-\frac{2}{x^2}\right) + \frac{2}{x} = 0$$

78. $y = x(\ln x) - 4x$

$$y' = x\left(\frac{1}{x}\right) + \ln x - 4 = -3 + \ln x$$

$$(x + y) - xy' = x + x\ln x - 4x - x(-3 + \ln x) = $$

79. $y = \frac{x^2}{2} - \ln x$

Domain: $x > 0$

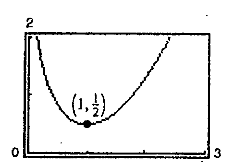
$$y' = x - \frac{1}{x}$$

$$= \frac{(x + 1)(x - 1)}{x}$$

$$= 0 \text{ when } x = 1.$$

$$y'' = 1 + \frac{1}{x^2} > 0$$

Relative minimum: $\left(1, \frac{1}{2}\right)$



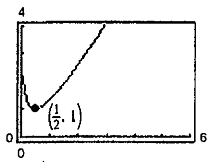
80. $y = 2x - \ln 2x = 2x - \ln 2 - \ln x$

Domain: $x > 0$

$$y' = 2 - \frac{1}{x} = \frac{2x - 1}{x} = 0 \text{ when } x = \frac{1}{2}.$$

$$y'' = \frac{1}{x^2} > 0$$

Relative minimum: $\left(\frac{1}{2}, 1\right)$



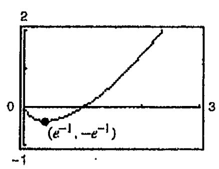
81. $y = x \ln x$

Domain: $x > 0$

$$y' = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x = 0 \text{ when } x = e^{-1}.$$

$$y'' = \frac{1}{x} > 0$$

Relative minimum: $\left(e^{-1}, -e^{-1}\right)$



82. $y = \dfrac{\ln x}{x}$

Domain: $x > 0$

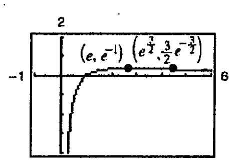
$$y' = \frac{x(1/x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4}$$

$$= \frac{2(\ln x) - 3}{x^3} = 0 \text{ when } x = e^{3/2}.$$

Relative maximum: $\left(e, e^{-1}\right)$

Point of inflection: $\left(e^{3/2}, \frac{3}{2}e^{-3/2}\right)$



83. $y = \dfrac{x}{\ln x}$

Domain: $0 < x < 1, x > 1$

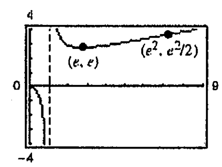
$$y' = \frac{(\ln x)(1) - (x)(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{(\ln x)^2(1/x) - (\ln x - 1)(2/x)\ln x}{(\ln x)^4}$$

$$= \frac{2 - \ln x}{x(\ln x)^3} = 0 \text{ when } x = e^2.$$

Relative minimum: (e, e)

Point of inflection: $\left(e^2, \frac{e^2}{2}\right)$



84. $y = x^2 \ln \dfrac{x}{4}$, Domain: $x > 0$

$$y' = x^2\left(\frac{1}{x}\right) + 2x \ln \frac{x}{4} = x\left(1 + 2 \ln \frac{x}{4}\right) = 0 \text{ when:}$$

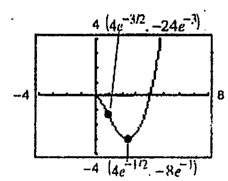
$$-1 = 2 \ln \frac{x}{4} \Rightarrow \ln \frac{x}{4} = -\frac{1}{2} \Rightarrow x = 4e^{-1/2}$$

$$y'' = 1 + 2 \ln \frac{x}{4} + 2x\left(\frac{1}{x}\right) = 3 + 2 \ln \frac{x}{4}$$

$$y'' = 0 \text{ when } x = 4e^{-3/2}$$

Relative minimum: $\left(4e^{-1/2}, -8e^{-1}\right)$

Point of inflection: $\left(4e^{-3/2}, -24e^{-3}\right)$



85. $f(x) = \ln x, \qquad f(1) = 0$

$f'(x) = \frac{1}{x}, \qquad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}, \qquad f''(1) = -1$

$P_1(x) = f(1) + f'(1)(x - 1) = x - 1, \qquad P_1(1) = 0$

$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2$

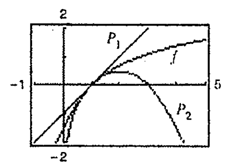
$= (x - 1) - \frac{1}{2}(x - 1)^2, \qquad P_2(1) = 0$

$P_1'(x) = 1, \qquad P_1'(1) = 1$

$P_2'(x) = 1 - (x - 1) = 2 - x, \qquad P_2'(1) = 1$

$P_2''(x) = -1, \qquad P_2''(1) = -1$

The values of f, P_1, P_2, and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



86. $f(x) = x \ln x, \qquad f(1) = 0$

$f'(x) = 1 + \ln x, \qquad f'(1) = 1$

$f''(x) = \frac{1}{x}, \qquad f''(1) = 1$

$P_1(x) = f(1) + f'(1)(x - 1) = x - 1, \qquad P_1(1) = 0$

$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2$

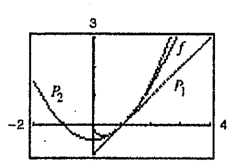
$= (x - 1) + \frac{1}{2}(x - 1)^2, \qquad P_2(1) = 0$

$P_1'(x) = 1, \qquad P_1'(1) = 1$

$P_2'(x) = 1 + (x - 1) = x, \qquad P_2'(1) = 1$

$P_2''(x) = x, \qquad P_2''(1) = 1$

The values of f, P_1, P_2, and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



87. Find x such that $\ln x = -x$.

$$f(x) = \ln x + x = 0$$

$$f'(x) = \frac{1}{x} + 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{1 - \ln x_n}{1 + x_n}\right]$$

n	1	2	3
x_n	0.5	0.5644	0.5671
$f(x_n)$	−0.1931	−0.0076	−0.0001

Approximate root: $x \approx 0.567$

88. Find x such that $\ln x = 3 - x$.

$$f(x) = x + (\ln x) - 3 = 0$$

$$f'(x) = 1 + \frac{1}{x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{4 - \ln x_n}{1 + x_n}\right]$$

n	1	2	3
x_n	2	2.2046	2.2079
$f(x_n)$	−0.3069	−0.0049	0.0000

Approximate root: $x \approx 2.208$

89.
$$y = x\sqrt{x^2 + 1}$$
$$\ln y = \ln x + \frac{1}{2}\ln(x^2 + 1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{x}{x^2 + 1}$$
$$\frac{dy}{dx} = y\left[\frac{2x^2 + 1}{x(x^2 + 1)}\right] = \frac{2x^2 + 1}{\sqrt{x^2 + 1}}$$

90.
$$y = \sqrt{x^2(x + 1)(x + 2)},\ x > 0$$
$$y^2 = x^2(x + 1)(x + 2)$$
$$2\ln y = 2\ln x + \ln(x + 1) + \ln(x + 2)$$
$$\frac{2}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{x + 1} + \frac{1}{x + 2}$$
$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{2}{x} + \frac{1}{x + 1} + \frac{1}{x + 2}\right]$$
$$\frac{dy}{dx} = \frac{\sqrt{x^2(x + 1)(x + 2)}}{2}\left[\frac{2(x + 1)(x + 2) + x(x + 2) + x(x + 1)}{x(x + 1)(x + 2)}\right] = \frac{4x^2 + 9x + 4}{2\sqrt{(x + 1)(x + 2)}}$$

91.
$$y = \frac{x^2\sqrt{3x - 2}}{(x + 1)^2}$$
$$\ln y = 2\ln x + \frac{1}{2}\ln(3x - 2) - 2\ln(x + 1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{x} + \frac{3}{2(3x - 2)} - \frac{2}{x + 1}$$
$$\frac{dy}{dx} = y\left[\frac{3x^2 + 15x - 8}{2x(3x - 2)(x + 1)}\right]$$
$$= \frac{3x^3 + 15x^2 - 8x}{2(x + 1)^3\sqrt{3x - 2}}$$

92.
$$y = \sqrt{\frac{x^2 - 1}{x^2 + 1}}$$
$$\ln y = \frac{1}{2}\left[\ln(x^2 - 1) - \ln(x^2 + 1)\right]$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{2}\left[\frac{2x}{x^2 - 1} - \frac{2x}{x^2 + 1}\right]$$
$$\frac{dy}{dx} = \sqrt{\frac{x^2 - 1}{x^2 + 1}}\left[\frac{2x}{x^4 - 1}\right]$$
$$= \frac{(x^2 - 1)^{1/2}2x}{(x^2 + 1)^{1/2}(x^2 - 1)(x^2 + 1)}$$
$$= \frac{2x}{(x^2 + 1)^{3/2}(x^2 - 1)^{1/2}}$$

93. $$y = \frac{x(x-1)^{3/2}}{\sqrt{x+1}}$$

$$\ln y = \ln x + \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{3}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)$$

$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{1}{x+1}\right]$$

$$= \frac{y}{2}\left[\frac{4x^2+4x-2}{x(x^2-1)}\right] = \frac{(2x^2+2x-1)\sqrt{x-1}}{(x+1)^{3/2}}$$

94. $$y = \frac{(x+1)(x-2)}{(x-1)(x+2)}$$

$$\ln y = \ln(x+1) + \ln(x-2) - \ln(x-1) - \ln(x+2)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x+1} + \frac{1}{x-2} - \frac{1}{x-1} - \frac{1}{x+2}$$

$$\frac{dy}{dx} = y\left[\frac{-2}{x^2-1} + \frac{4}{x^2-4}\right] = y\left[\frac{2x^2+4}{(x^2-1)(x^2-4)}\right]$$

$$= \frac{(x+1)(x+2)}{(x-1)(x-2)} \cdot \frac{2x^2+4}{(x+1)(x-1)(x+2)(x-2)}$$

$$= \frac{2(x^2+2)}{(x-1)^2(x-2)^2}$$

95. The domain of the natural logarithmic function is $(0, \infty)$ and the range is $(-\infty, \infty)$. The function is continuous, increasing, and one-to-one, and its graph is concave downward. In addition, if a and b are positive numbers and n is rational, then $\ln(1) = 0$, $\ln(a \cdot b) = \ln a + \ln b$, $\ln(a^n) = n \ln a$, and $\ln(a/b) = \ln a - \ln b$.

96. The base of the natural logarithmic function is e.

97. $g(x) = \ln f(x), \quad f(x) > 0$

$$g'(x) = \frac{f'(x)}{f(x)}$$

(a) Yes. If the graph of g is increasing, then $g'(x) > 0$. Because $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and so, $f'(x) > 0$. Therefore, the graph of f is increasing.

(b) No. Let $f(x) = x^2 + 1$ (positive and concave up). $g(x) = \ln(x^2+1)$ is not concave up.

98. (a) $\lim_{h\to\infty} T = 20$

The temperature of the object seems to approach 20°C, which is the temperature of the surrounding medium.

(b) The temperature changes most rapidly when it is first removed from the furnace. The slope is steepest at $h = 0$.

99. False

$\ln x + \ln 25 = \ln(25x) \neq \ln(x + 25)$

100. False. The property is $\ln xy = \ln x + \ln y$ (for $x, y > 0$). As a counter example, let $x = y = e$. Then

$\ln xy = \ln e^2 = 2 \quad$ and $\quad \ln x \ln y = 1 \cdot 1 = 1.$

101. False; π is a constant.

$$\frac{d}{dx}[\ln \pi] = 0$$

102. False. If $y = \ln e = 1$, then $y' = 0$.

103. $t = 13.375 \ln\left(\frac{x}{x-1250}\right), \quad x > 1250$

(a)

(b) When $x = 1398.43$: $t \approx 30$ years

Total amount paid $= (1398.43)(30)(12) = \$503{,}434.80$

(c) When $x = 1611.19$: $t \approx 20$ years

Total amount paid $= (1611.19)(20)(12) = \$386{,}685.60$

(d) $\dfrac{dt}{dx} = \dfrac{d}{dx}\left[13.375\left(\ln x - \ln(x - 1250)\right)\right] = 13.375\left[\dfrac{1}{x} - \dfrac{1}{x - 1250}\right] = \dfrac{-16718.75}{x(x - 1250)}$

When $x = 1398.43$: $\dfrac{dt}{dx} \approx -0.0805$

When $x = 1611.19$: $\dfrac{dt}{dx} \approx -0.0287$

(e) The benefits include a shorter term, and a lower total amount paid.

104. (a) $\beta = \dfrac{10}{\ln 10}\ln\left(\dfrac{I}{10^{-16}}\right)$

$= \dfrac{10}{\ln 10}\left[\ln I - \ln 10^{-16}\right]$

$= \dfrac{10}{\ln 10}\left[\ln I + 16\ln 10\right]$

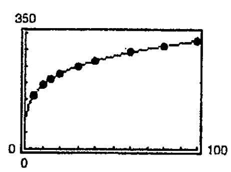
$= \dfrac{10}{\ln 10}\ln I + 160$

$= 10\log_{10} I + 160$

(b) $\beta\left(10^{-5}\right) = \dfrac{10}{\ln 10}\ln 10^{-5} + 160$

$= -50 + 160 = 110$ decibels

105. (a)

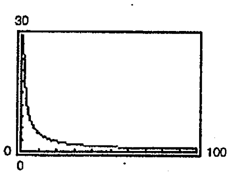


(b) $T'(p) = \dfrac{34.96}{p} + \dfrac{3.955}{\sqrt{p}}$

$T'(10) \approx 4.75$ deg/lb/in.^{2}

$T'(70) \approx 0.97$ deg/lb/in.^{2}

(c)



$\lim\limits_{p\to\infty} T'(p) = 0$

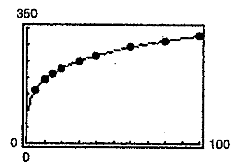
Answers will vary. *Sample answer*: As the pounds per square inch approach infinity, the temperature will not change.

106. (a) You get an error message because $\ln h$ does not exist for $h = 0$.

(b) Reversing the data, you obtain $h = 0.8627 - 6.4474 \ln p$.

[**Note:** Fit a line to the data $(x, y) = (\ln p, h)$.]

(c)



(d) If $p = 0.75$, $h \approx 2.72$ km.

(e) If $h = 13$ km, $p \approx 0.15$ atmosphere.

(f) $h = 0.8627 - 6.4474 \ln p$

$1 = -6.4474\dfrac{1}{p}\dfrac{dp}{dh}$ (implicit differentiation)

$\dfrac{dp}{dh} = \dfrac{p}{-6.4474}$

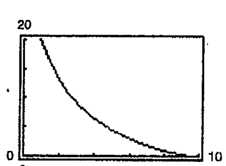
For $h = 5$, $p = 0.5264$ and $dp/dh = -0.0816$ atmos/km.

For $h = 20$, $p = 0.0514$ and $dp/dh = -0.0080$ atmos/km.

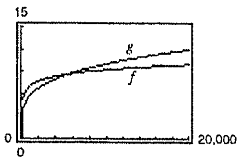
As the altitude increases, the rate of change of pressure decreases.

107. $y = 10\ln\left(\dfrac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2} = 10\left[\ln\left(10 + \sqrt{100 - x^2}\right) - \ln x\right] - \sqrt{100 - x^2}$

(a)



(b) $f(x) = \ln x, g(x) = \sqrt[4]{x}$



$$f'(x) = \frac{1}{x},\ g'(x) = \frac{1}{4\sqrt[4]{x^3}}$$

For $x > 256$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for "large" values of x. $f(x) = \ln x$ increases very slowly for "large" values of x.

Section 5.2 The Natural Logarithmic Function: Integration

1. $\int \frac{5}{x}\,dx = 5\int \frac{1}{x}\,dx = 5\ln|x| + C$

2. $\int \frac{10}{x}\,dx = 10\int \frac{1}{x}\,dx = 10\ln|x| + C$

3. $u = x + 1, du = dx$

$$\int \frac{1}{x+1}\,dx = \ln|x + 1| + C$$

4. $u = x - 5, du = dx$

$$\int \frac{1}{x-5}\,dx = \ln|x - 5| + C$$

5. $u = 2x + 5, du = 2\,dx$

$$\int \frac{1}{2x+5}\,dx = \frac{1}{2}\int \frac{1}{2x+5}(2)\,dx = \frac{1}{2}\ln|2x + 5| + C$$

6. $u = 5 - 4x, du = -4\,dx$

$$\int \frac{9}{5-4x}\,dx = -\frac{9}{4}\int \frac{1}{5-4x}(-4dx) = -\frac{9}{4}\ln|5 - 4x| + C$$

7. $u = x^2 - 3, du = 2x\,dx$

$$\int \frac{x}{x^2-3}\,dx = \frac{1}{2}\int \frac{1}{x^2-3}(2x)\,dx = \frac{1}{2}\ln|x^2 - 3| + C$$

8. $u = 5 - x^3, du = -3x^2\,dx$

$$\int \frac{x^2}{5-x^3}\,dx = -\frac{1}{3}\int \frac{1}{5-x^3}(-3x^2)\,dx = -\frac{1}{3}\ln|5 - x^3| + C$$

9. $u = x^4 + 3x, du = (4x^3 + 3)\,dx$

$$\int \frac{4x^3+3}{x^4+3x}\,dx = \int \frac{1}{x^4+3x}(4x^3+3)\,dx = \ln|x^4 + 3x| + C$$

10. $u = x^3 - 3x^2, du = (3x^2 - 6x)\,dx = 3(x^2 - 2x)\,dx$

$$\int \frac{x^2-2x}{x^3-3x^2}\,dx = \frac{1}{3}\int \frac{1}{x^3-3x^2}(3x^2-6x)\,dx = \frac{1}{3}\ln|x^3 - 3x^2| + C$$

11. $\int \frac{x^2-4}{x}\,dx = \int\left(x - \frac{4}{x}\right)dx$

$$= \frac{x^2}{2} - 4\ln|x| + C$$
$$= \frac{x^2}{2} - \ln(x^4) + C$$

12. $\int \frac{x^3-8x}{x^2}\,dx = \int\left(x - \frac{8}{x}\right)dx$

$$= \frac{x^2}{2} - 8\ln|x| + C$$

13. $u = x^3 + 3x^2 + 9x, du = 3(x^2 + 2x + 3)\,dx$

$$\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx = \frac{1}{3}\int \frac{3(x^2+2x+3)}{x^3+3x^2+9x}\,dx = \frac{1}{3}\ln|x^3 + 3x^2 + 9x| + C$$