Calculus Ch. 5.1b, 5.3 Notes

Warm-up Problem: If $g(2) = 3$ and $g'(2) = -4$, find $f'(2)$ if $f(x) = x^2\ln(g(x))$.

Logarithmic Differentiation : Simplifying non-log functions using log properties to expand before differentiating.

Log differentiation steps:

1. Take the **ln** (natural log) of both sides
2. Simplify and expand using log properties
3. Use implicit differentiation
4. Substitute for y

Example 1: Find the derivative of $y = \dfrac{(x-2)^2}{\sqrt{x^2+1}}$

Example 2: Find the derivative of $y = x^{2x+3}$

Absolute Value Rule: $\dfrac{d}{dx}\ln|u| = \dfrac{u'}{u}$

Example 3: Find $\frac{d}{dx}\ln|x^2-5|$

Example 4: Locate any relative extrema and inflection points for $y = \dfrac{x}{\ln x}$

Ch.5.3 Notes Derivatives of Inverse Function

A. Inverse Functions

1) x's and y's are swapped
2) Domains and ranges are swapped
3) Graphs are symmetric over the line $y = x$
4) F(x) must be one-to-one (pass the horizontal line test) for its inverse to be a function
5) A function that is **monotonic** (always increasing or always decreasing) will always have an inverse that is a function
6) If f and g are inverses, then $f(g(x)) = x$ and $g(f(x)) = x$

Example 4: find the inverse of $f(x) = 6x + 2$

*ALWAYS restrict the domain of the inverse function to the range of the original function

Ex. 5 $f(x) = \sqrt{x-5}$. Find the domain of the inverse function

Evaluate derivative of inverse at a point: (find $(f^{-1})'(a)$

$f\,(b) = a$	$(f^{-1})(a) = b$
$f\,'(b) = n$	$(f^{-1})'(a) = \frac{1}{n}$

Example 6: $f(x) = x^3 + 4x + 2$ find $(f^{-1})'(-3)$

Example 7: $f(x) = \sqrt{x^3 - 7}$ find $(f^{-1})'(1)$

Example 8: If $g(f(x)) = x$, $g(7) = 2$, and $g'(7) = 10$, then $f'(2)$ is

Example 9: If $g(f(x)) = x$, $g(9) = 3$, and $g'(9) = -4$, then $f'(3)$ is

Calculus Ch. 5.1b, 5.3 Notes

Recall $\frac{d}{dx} \ln u = \frac{u'}{u}$

Warm-up Problem: If g(2) = 3 and g ' (2) = -4, find f ' (2) if f(x) = x^2ln(g(x)).

$$f'(x) = 2x \cdot \ln(g(x)) + 2^2 \cdot \frac{g'(x)}{g(x)}$$

$$f'(2) = 2(2) \cdot \ln[g(2)] + 2^2 \cdot \frac{g'(2)}{g(2)}$$

$$f'(2) = 4 \cdot \ln(3) + 2^2(-4/3)$$

$$\boxed{f'(2) = 4\ln 3 - 16/3}$$

Logarithmic Differentiation : Simplifying non-log functions using log properties to expand before differentiating.

Log differentiation steps:

1. Take the **ln** (natural log) of both sides
2. Simplify and expand using log properties
3. Use implicit differentiation
4. Substitute for y

Example 1: Find the derivative of $y = \frac{(x-2)^2}{\sqrt{x^2+1}}$

$$\ln y = \ln\left[\frac{(x-2)^2}{(x^2+1)^{1/2}}\right]$$

$$\ln y = \ln(x-2)^2 - \ln(x^2+1)^{1/2}$$

$$\ln y = 2\ln(x-2) - \frac{1}{2}\ln(x^2+1)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\left(\frac{1}{x-2}\right) - \frac{1}{2}\left(\frac{2x}{x^2+1}\right)$$

$$\frac{dy}{dx} = y\left[\frac{2}{x-2} - \frac{x}{x^2+1}\right]$$

$$\frac{dy}{dx} = \frac{(x-2)^2}{\sqrt{x^2+1}}\left[\frac{2}{x-2} - \frac{x}{x^2+1}\right]$$

Example 2: Find the derivative of y = x^{2x+3}

$$\ln y = \ln x^{2x+3}$$

$$\ln y = (2x+3)(\ln x)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = (2)[\ln x] + (2x+3)\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = y\left[2\ln x + \frac{2x+3}{x}\right]$$

$$\frac{dy}{dx} = x^{2x+3}\left[2\ln x + 2 + \frac{3}{x}\right]$$

Absolute Value Rule: $\frac{d}{dx}\ln|u| = \frac{u'}{u}$

Example 3: Find $\frac{d}{dx}\ln|x^2-5| = \frac{2x}{x^2-5}$

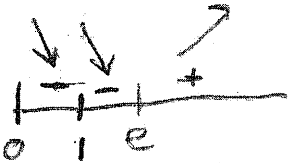
Example 4: Locate any relative extrema and inflection points for $y = \frac{x}{\ln x}$

$$y'(x) = \frac{1\ln x - x\left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} \rightarrow$$

set $\ln x - 1 = 0$

$\ln x = 1$

$e^1 = x$



Rel. min at $\left(e, \frac{e}{\ln e}\right)$

$= (e, e)$

$$y''(x) = \frac{\left(\frac{1}{x}\right)(\ln x)^2 - (\ln x - 1)\,2(\ln x)\left(\frac{1}{x}\right)}{(\ln x)^4}$$

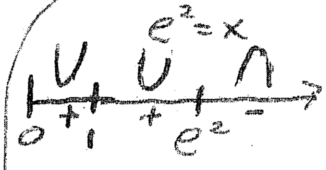
$$= \frac{\frac{1}{x}\ln x\left[\ln x - 2\ln x + 2\right]}{(\ln x)^4}$$

$$y''(x) = \frac{-\ln x + 2}{x(\ln x)^3}$$

$2 - \ln x = 0$

$2 = \ln x$

$e^2 = x$



POI at $\left(e^2, \frac{e^2}{2}\right)$ b/c $y''(x)$ change signs.

Ch.5.3 Notes Derivatives of Inverse Function

A. Inverse Functions

1) x's and y's are swapped
2) Domains and ranges are swapped
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4) F(x) must be one-to-one (pass the horizontal line test) for its inverse to be a function
5) A function that is **monotonic** (always increasing or always decreasing) will always have an inverse that is a function
6) If f and g are inverses, then f(g(x)) = x and g(f(x)) = x

Example 4: find the inverse of f(x) = 6x + 2

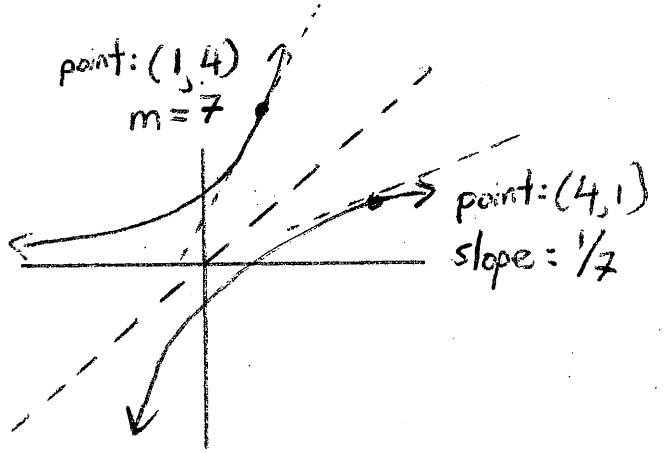
*ALWAYS restrict the domain of the inverse function to the range of the original function

Ex. 5 f(x) = $\sqrt{x-5}$. Find the domain of the inverse function

* At their corresponding points, the slopes of tangent line will be reciprocals of each other

Evaluate derivative of inverse at a point: (find $(f^{-1})'(a)$

$f\,(b) = a$	$(f^{-1})(a) = b$
$f\,'(b) = n\cdot$	$(f^{-1})'(a) = \frac{1}{n}$



Example 6: f(x) = $x^3 + 4x + 2$ find $(f^{-1})'(-3)$ $g'(-3)$

$f(-1)=-3$	$g(-3)=\underline{-1}$
$f'(-1)=7$	$g'(3)=\frac{1}{7}$

$$\boxed{g'(3)=\frac{1}{7}}$$

$$-3=x^3+4x+2$$
$$x=-1$$

$$f'(x)=3x^2+4$$
$$f'(-1)=3(-1)^2+4$$
$$=3+4=7$$

Example 7: f(x) = $\sqrt{x^3-7}$ find $(f^{-1})'(1)$ Find $g'(1)$

$f(2)=1$	$g(1)=\underline{2}$
$f'(2)=6$	$g'(1)=\frac{1}{6}$

$$1=\sqrt{x^3-7}$$
$$1=x^3-7 \quad x^3=8, x=2$$

$$f(x)=(x^3-7)^{1/2}$$
$$f'(x)=\frac{1}{2}(x^3-7)^{-1/2}(3x^2)$$
$$=\frac{3x^2}{2\sqrt{x^3-7}}$$

$$f'(2)=\frac{12}{2(1)} \longleftarrow$$

Example 8: If g(f(x)) = x , g(7) = 2, and g'(7) = 10, then f'(2) is

$g(7)=2$	$f(2)=7$
$g'(7)=10$	$f'(2)=\frac{1}{10}$

$$\boxed{f'(2)=\frac{1}{10}}$$

Example 9: If g(f(x)) = x , g(9) = 3, and g'(9) = -4, then f'(3) is

$g(9)=3$	$f(3)=9$
$g'(9)=-4$	$f'(3)=\frac{-1}{4}$

$$\boxed{f'(3)=-\frac{1}{4}}$$

Log Differentiation:
Use log differentiation to find dy/dx

Log Properties

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

89. $y = x\sqrt{x^2 + 1}, \quad x > 0$

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90. $y = \sqrt{x^2(x + 1)(x + 2)}, \quad x > 0$

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91. $y = \dfrac{x^2\sqrt{3x - 2}}{(x + 1)^2}, \quad x > \dfrac{2}{3}$

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92. $y = \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}, \quad x > 1$

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93. $y = \dfrac{x(x - 1)^{3/2}}{\sqrt{x + 1}}, \quad x > 1$

5.3 Classwork Inverse Functions and Finding Derivative of Inverse at a Point

Finding an Inverse Function **In Exercises 35–46, (a) find the inverse function of f, (b) graph f and f^{-1} on the same set of coordinate axes, (c) describe the relationship between the graphs, and (d) state the domain and range of f and f^{-1}.**

35. $f(x) = 2x - 3$

36. $f(x) = 7 - 4x$

37. $f(x) = x^5$

38. $f(x) = x^3 - 1$

39. $f(x) = \sqrt{x}$

40. $f(x) = x^2, \quad x \geq 0$

41. $f(x) = \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$

42. $f(x) = \sqrt{x^2 - 4}, \quad x \geq 2$

43. $f(x) = \sqrt[3]{x - 1}$

44. $f(x) = x^{2/3}, \quad x \geq 0$

Evaluating the Derivative of an Inverse Function

Exercises 63–70, verify that f has an inverse. Then use function f and the given real number a to find $(f^{-1})'(a)$. (H

63. $f(x) = 5 - 2x^3, \quad a = 7$

64. $f(x) = x^3 + 2x - 1, \quad a = 2$

65. $f(x) = \frac{1}{27}(x^5 + 2x^3), \quad a = -11$

66. $f(x) = \sqrt{x - 4}, \quad a = 2$

Log Differentiation:
Use log differentiation to find dy/dx

1) Take "ln" of both sides
2) Expand using log properties
3) Take derivative (implicit diff. on left side)

Log Properties

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

89. $y = x\sqrt{x^2+1}, \quad x > 0$

$$\ln y = \ln\left[x\sqrt{x^2+1}\right]$$
$$\ln y = \ln x + \ln\sqrt{x^2+1}$$
$$\ln y = \ln x + \ln(x^2+1)^{1/2}$$
$$\ln y = \ln x + \tfrac{1}{2}\ln(x^2+1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1}$$
$$\frac{dy}{dx} = y\left[\frac{1}{x} + \frac{x}{x^2+1}\right]$$
$$\boxed{\frac{dy}{dx} = \left[x\sqrt{x^2+1}\right]\left[\frac{1}{x} + \frac{x}{x^2+1}\right.}$$

90. $y = \sqrt{x^2(x+1)(x+2)}, \quad x > 0$

$$\ln y = \ln\left(x^2(x+1)(x+2)\right)^{1/2}$$
$$\ln y = \frac{1}{2}\ln\left[x^2(x+1)(x+2)\right]$$
$$\ln y = \tfrac{1}{2}\ln x^2 + \tfrac{1}{2}\ln(x+1) + \tfrac{1}{2}(x+2)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{2}\cdot\frac{2x}{x^2} + \frac{1}{2}\cdot\frac{1}{x+1} + \frac{1}{2}\cdot\frac{1}{x+2}$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{2(x+2)}$$
$$\frac{dy}{dx} = y\left[\frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{2(x+2)}\right] = \boxed{\sqrt{x^2(x+1)(x+2)}\left[\frac{1}{x} + \frac{1}{2(x+1)} + \right.}$$

91. $y = \dfrac{x^2\sqrt{3x-2}}{(x+1)^2}, \quad x > \dfrac{2}{3}$

$$\ln y = \ln\left[\frac{x^2(3x-2)^{1/2}}{(x+1)^2}\right]$$
$$\ln y = \ln x^2 + \ln(3x-2)^{1/2} - \ln(x+1)^2$$
$$\ln y = 2\ln x + \tfrac{1}{2}\ln(3x-2) - 2\ln(x+1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = 2\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{3}{3x-2} - 2\cdot\frac{1}{x+1}$$
$$\frac{dy}{dx} = y\left[\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x+1}\right] = \boxed{\frac{x^2\sqrt{3x-2}}{(x+1)^2}\left[\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x+1}\right]}$$

92. $y = \sqrt{\dfrac{x^2-1}{x^2+1}}, \quad x > 1$

$$\ln y = \ln\left(\frac{x^2-1}{x^2+1}\right)^{1/2}$$
$$\ln y = \frac{1}{2}\ln\left(\frac{x^2-1}{x^2+1}\right)$$
$$\ln y = \tfrac{1}{2}\ln(x^2-1) - \tfrac{1}{2}\ln(x^2+1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{2}\cdot\frac{2x}{x^2-1} - \frac{1}{2}\cdot\frac{2x}{x^2+1}$$
$$\frac{dy}{dx} = y\cdot\left[\frac{x}{x^2-1} - \frac{x}{x^2+1}\right]$$
$$\boxed{\frac{dy}{dx} = \sqrt{\frac{x^2-1}{x^2+1}}\left[\frac{x}{x^2-1} - \frac{x}{x^2+1}\right]}$$

93. $y = \dfrac{x(x-1)^{3/2}}{\sqrt{x+1}}, \quad x > 1$

$$\ln y = \ln\left[\frac{x(x-1)^{3/2}}{(x+1)^{1/2}}\right]$$
$$\ln y = \ln x + \ln(x-1)^{3/2} - \ln(x+1)^{1/2}$$
$$\ln y = \ln x + \tfrac{3}{2}\ln(x-1) - \tfrac{1}{2}\ln(x+1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{3}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+1}$$
$$\frac{dy}{dx} = y\left[\frac{1}{x} + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}\right]$$
$$\boxed{\frac{dy}{dx} = \frac{x(x-1)^{3/2}}{\sqrt{x+1}}\left[\frac{1}{x} + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}\right]}$$

5.3 Classwork Inverse Functions and Finding Derivative of Inverse at a Point

Finding an Inverse Function In Exercises 35–46, (a) find the inverse function of f, (b) graph f and f^{-1} on the same set of coordinate axes, (c) describe the relationship between the graphs, and (d) state the domain and range of f and f^{-1}.

To find inverse function:

1) switch x and y
2) Solve for new y.

35. $f(x) = 2x - 3$

$y = 2x - 3$

$x = 2y - 3$

$x + 3 = 2y$

$\frac{x+3}{2} = y$

$\boxed{f^{-1}(x) = \frac{x+3}{2}}$

36. $f(x) = 7 - 4x$

$y = 7 - 4x$

$x = 7 - 4y$

$4y = 7 - x$

$y = \frac{7-x}{4}$

$\boxed{f^{-1}(x) = \frac{7-x}{4}}$

37. $f(x) = x^5$

$y = x^5$

$x = y^5$

$y^5 = x$

$y = \sqrt[5]{x}$

$\boxed{f^{-1}(x) = \sqrt[5]{x}}$

38. $f(x) = x^3 - 1$

$y = x^3 - 1$

$x = y^3 - 1$

$x + 1 = y^3$

$\sqrt[3]{x+1} = y$

$\boxed{f^{-1}(x) = \sqrt[3]{x+1}}$

39. $f(x) = \sqrt{x}$

$y = \sqrt{x}$

$x = \sqrt{y}$

$x^2 = y$

$\boxed{f^{-1}(x) = x^2}$

40. $f(x) = x^2, \quad x \geq 0$

$y = x^2$

$x = y^2$

$\sqrt{x} = y$

$\boxed{f^{-1}(x) = \sqrt{x}}$

41. $f(x) = \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$

$y = \sqrt{4 - x^2}$

$x = \sqrt{4 - y^2}$

$x^2 = 4 - y^2$

$y^2 = 4 - x^2$

$y = \pm\sqrt{4 - x^2}$

$\boxed{f^{-1}(x) = \sqrt{4 - x^2}}$

42. $f(x) = \sqrt{x^2 - 4}, \quad x \geq 2$

$y = \sqrt{x^2 - 4}$

$x = \sqrt{y^2 - 4}$

$x^2 = y^2 - 4$

$x^2 + 4 = y^2$

$y = \sqrt{x^2 + 4}$

$\boxed{f^{-1}(x) = \sqrt{x^2 + 4}}$

43. $f(x) = \sqrt[3]{x - 1}$

$y = \sqrt[3]{x - 1}$

$x = \sqrt[3]{y - 1}$

$x^3 = y - 1$

$x^3 + 1 = y$

$\boxed{f^{-1}(x) = x^3 + 1}$

44. $f(x) = x^{2/3}, \quad x \geq 0$

$y = x^{2/3}$

$x = y^{2/3}$

$(x)^{3/2} = (y^{2/3})^{3/2}$

$x^{3/2} = y$

$\boxed{f^{-1}(x) = x^{3/2}}$

Evaluating the Derivative of an Inverse Function

Exercises 63–70, verify that f has an inverse. Then use function f and the given real number a to find $(f^{-1})'(a)$. (H

$f(b)=a$	$f^{-1}(a)=b$
$f'(b)=n$	$f^{-1}{}'(a)=\frac{1}{n}$

63. $f(x) = 5 - 2x^3, \quad a = 7$

$f(-1)=7$	$f^{-1}(7)=\underline{-1}$
$f'(-1)=\underline{-6}$	$(f^{-1})'(7)=\boxed{\frac{-1}{6}}$

$7=5-2x^3$

$2=-2x^3$

$-1=x^3$

$\underline{\underline{-1=x}}$

$f(x)=5-2x^3$

$f'(x)=-6x^2$

$f'(-1)=-6(-1)^2$

$=-6$

64. $f(x) = x^3 + 2x - 1, \quad a = 2$

$f(\underline{1})=2$	$f^{-1}(2)=\underline{1}$
$f'(1)=\underline{5}$	$(f^{-1})'(2)=\boxed{\frac{1}{5}}$

$2=x^3+2x-1$

$0=x^3+2x-3$

$\underline{\underline{x=1}}$

$f(x)=x^3+2x-1$

$f'(x)=3x^2+2$

$f'(1)=3(1)^2+2=5$

65. $f(x) = \frac{1}{27}(x^5 + 2x^3), \quad a = -11$

$f(-3)=-11$	$f^{-1}(-11)=\underline{-3}$
$f'(-3)=\underline{17}$	$(f^{-1})'(-11)=\boxed{\frac{1}{17}}$

$-11=\frac{1}{27}(x^5+2x^3)$

$-297=x^5+2x^3$

$x=-3$

$f(x)=\frac{1}{27}(x^5+2x^3)$

$f'(x)=\frac{1}{27}(5x^4+6x^2)$

$f'(-3)=\frac{1}{27}(5(-3)^4+6(-3)^2)$

$=\frac{1}{27}(405+54)=\frac{459}{27}$

$=17$

66. $f(x) = \sqrt{x - 4}, \quad a = 2$

$f(8)=\underline{2}$	$(f^{-1})(2)=\underline{8}$
$f'(8)=\underline{\frac{1}{4}}$	$(f^{-1})'(2)=\boxed{4}$

$2=\sqrt{x-4}$

$4=x-4$

$\underline{\underline{8=x}}$

$f(x)=\sqrt{x-4}$

$f(x)=(x-4)^{1/2}$

$f'(x)=\frac{1}{2}(x-4)^{-1/2}(1)$

$f'(8)=\frac{1}{2}(8-4)^{-1/2}$

$=\frac{1}{2}(4)^{-1/2}=\frac{1}{2}\left(\frac{1}{\sqrt{4}}\right)=\frac{1}{2}\left(\frac{1}{2}\right)=\frac{\ }{4}$