Ch. 5.1b Natural Log Differentiation Classwork

Finding a Derivative In Exercises 41–64, find the derivative of the function.

$$\frac{d}{dx}[\ln u] = \frac{u'}{u}$$

Reminder: Expand log expression fully before differentiating

Log Properties

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

41. $f(x) = \ln(3x)$

42. $f(x) = \ln(x - 1)$

43. $g(x) = \ln x^2$

44. $h(x) = \ln(2x^2 + 1)$

45. $y = (\ln x)^4$

46. $y = x^2 \ln x$

47. $y = \ln(t + 1)^2$

48. $y = \ln\sqrt{x^2 - 4}$

49. $y = \ln\left(x\sqrt{x^2 - 1}\right)$

50. $y = \ln[t(t^2 + 3)^3]$

51. $f(x) = \ln\left(\frac{x}{x^2 + 1}\right)$

52. $f(x) = \ln\left(\frac{2x}{x + 3}\right)$

53. $g(t) = \frac{\ln t}{t^2}$

54. $h(t) = \frac{\ln t}{t}$

55. $y = \ln(\ln x^2)$

56. $y = \ln(\ln x)$

57. $y = \ln\sqrt{\frac{x + 1}{x - 1}}$

58. $y = \ln\sqrt[3]{\frac{x - 1}{x + 1}}$

59. $f(x) = \ln\left(\frac{\sqrt{4 + x^2}}{x}\right)$

60. $f(x) = \ln\left(x + \sqrt{4 + x^2}\right)$

Finding a Derivative Implicitly **In Exercises 73–76, use implicit differentiation to find dy/dx.**

73. $x^2 - 3 \ln y + y^2 = 10$

74. $\ln xy + 5x = 30$

75. $4x^3 + \ln y^2 + 2y = 2x$

76. $4xy + \ln x^2y = 7$

Log Differentiation:
Use log differentiation to find dy/dx

Log Properties

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

89. $y = x\sqrt{x^2 + 1}, \quad x > 0$

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90. $y = \sqrt{x^2(x + 1)(x + 2)}, \quad x > 0$

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91. $y = \dfrac{x^2\sqrt{3x - 2}}{(x + 1)^2}, \quad x > \dfrac{2}{3}$

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92. $y = \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}, \quad x > 1$

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93. $y = \dfrac{x(x - 1)^{3/2}}{\sqrt{x + 1}}, \quad x > 1$

Key

Ch. 5.1b Natural Log Differentiation Classwork

Finding a Derivative In Exercises 41–64, find the derivative of the function.

$$\frac{d}{dx}[\ln u] = \frac{u'}{u}$$

*Cannot expand: $\ln(a-b)$ and $\ln(a+b)$

Reminder: Expand log expression fully before differentiating

Log Properties

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

41. $f(x) = \ln(3x)$

$f'(x) = \frac{3}{3x} = \boxed{\frac{1}{x}}$

or $y = \ln 3 + \ln x$

$\boxed{y' = 0 + \frac{1}{x}}$

42. $f(x) = \ln(x - 1)$

$f'(x) = \boxed{\frac{1}{x-1}}$

43. $g(x) = \ln x^2$

$g(x) = 2\ln x$

$g'(x) = 2\left(\frac{1}{x}\right) = \boxed{\frac{2}{x}}$

44. $h(x) = \ln(2x^2 + 1)$

$h'(x) = \boxed{\frac{4x}{2x^2+1}}$

45. $y = (\ln x)^4$

*chain Rule: $\frac{d}{dx} f[g(x)] = f'[g(x)] \cdot g'(x)$

$y' = 4[\ln x]^3 \cdot \left(\frac{1}{x}\right)$

$\boxed{y' = \frac{4(\ln x)^3}{x}}$

46. $y = x^2 \ln x$

*product rule: $\frac{d}{dx} f(x) \cdot g(x) = f'(x)g(x) + f(x)g'(x)$

$y' = \underbrace{2x}_{f'} \cdot \underbrace{\ln x}_{g} + \underbrace{x^2}_{f} \cdot \underbrace{\frac{1}{x}}_{g'}$

$y' = 2x\ln x + \frac{x^2}{x} = \boxed{2x\ln x + x}$

47. $y = \ln(t + 1)^2$

$y = 2\ln(t+1)$

$y' = 2 \cdot \frac{1}{t+1} = \boxed{\frac{2}{t+1}}$

48. $y = \ln\sqrt{x^2 - 4}$

$y = \ln(x^2-4)^{1/2}$

$y = \frac{1}{2}\ln(x^2-4)$

$y' = \frac{1}{2} \cdot \frac{2x}{x^2-4}$

$\boxed{y' = \frac{x}{x^2-4}}$

49. $y = \ln\left(x\sqrt{x^2 - 1}\right)$

$y = \ln\left(x \cdot (x^2-1)^{1/2}\right)$

$y = \ln x + \ln(x^2-1)^{1/2}$

$y = \ln x + \frac{1}{2}\ln(x^2-1)$

$y' = \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2-1}$

$\boxed{y' = \frac{1}{x} + \frac{x}{x^2-1}}$

50. $y = \ln[t(t^2 + 3)^3]$

$y = \ln t + \ln(t^2+3)^3$

$y = \ln t + 3\ln(t^2+3)$

$y' = \frac{1}{t} + 3 \cdot \frac{2t}{t^2+3}$

$\boxed{y' = \frac{1}{t} + \frac{6t}{t^2+3}}$

51. $f(x) = \ln\left(\frac{x}{x^2+1}\right)$

$f(x) = \ln x - \ln(x^2+1)$

$\boxed{f'(x) = \frac{1}{x} - \frac{2x}{x^2+1}}$

52. $f(x) = \ln\left(\frac{2x}{x+3}\right)$

$f(x) = \ln(2x) - \ln(x+3)$

$f'(x) = \frac{2}{2x} - \frac{1}{x+3} = \boxed{\frac{1}{x} - \frac{1}{x+3}}$

53. $g(t) = \frac{\ln t}{t^2}$

*quotient rule : $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$g'(t) = \frac{(\frac{1}{t})(t^2) - (\ln t)(2t)}{[t^2]^2}$

$= \frac{t - 2t\ln t}{t^4}$

$= \frac{\cancel{t}(1-2\ln t)}{t^{\cancel{4}\,3}}$

$= \boxed{\frac{1-2\ln t}{t^3}}$

54. $h(t) = \frac{\ln t}{t}$

*quotient rule

$h'(t) = \frac{(\frac{1}{t})(t) - (\ln t)(1)}{t^2}$

$\boxed{h'(t) = \frac{1-\ln t}{t^2}}$

55. $y = \ln(\ln x^2)$

*chain rule

$y = \ln[2\ln x]$

$y' = \frac{\cancel{2}(\frac{1}{x})}{\cancel{2}\ln x} = \frac{\frac{1}{x} \cdot \frac{1}{\ln x}}{\ln x \cdot \frac{1}{\ln x}}$

$\boxed{y' = \frac{1}{x\ln x}}$

56. $y = \ln(\ln x)$

*chain rule

$y' = \frac{\frac{1}{x} \cdot \frac{1}{\ln x}}{\ln x \cdot \frac{1}{\ln x}}$

$\boxed{y' = \frac{1}{x\ln x}}$

57. $y = \ln\sqrt{\frac{x+1}{x-1}}$

$y = \ln\left(\frac{x+1}{x-1}\right)^{1/2}$

$y = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$

$y = \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x-1)$

$y' = \frac{1}{2} \cdot \frac{1}{x+1} - \frac{1}{2} \cdot \frac{1}{x-1}$

$\boxed{y' = \frac{1}{2(x+1)} - \frac{1}{2(x-1)}}$

58. $y = \ln\sqrt[3]{\frac{x-1}{x+1}}$

$y = \ln\left(\frac{x-1}{x+1}\right)^{1/3}$

$y = \frac{1}{3}\ln\left(\frac{x-1}{x+1}\right)$

$y = \frac{1}{3}\ln(x-1) - \frac{1}{3}\ln(x+1)$

$y' = \frac{1}{3} \cdot \frac{1}{x-1} - \frac{1}{3} \cdot \frac{1}{x+1}$

$\boxed{y' = \frac{1}{3(x-1)} - \frac{1}{3(x+1)}}$

59. $f(x) = \ln\left(\frac{\sqrt{4+x^2}}{x}\right)$

$f(x) = \ln\sqrt{4+x^2} - \ln x$

$f(x) = \ln(4+x^2)^{1/2} - \ln x$

$f(x) = \frac{1}{2}\ln(4+x^2) - \ln x$

$f'(x) = \frac{1}{2} \cdot \frac{2x}{4+x^2} - \frac{1}{x}$

$\boxed{f'(x) = \frac{x}{4+x^2} - \frac{1}{x}}$

60. $f(x) = \ln(x + \sqrt{4+x^2})$

*cannot expand!! $f(x) = \ln[x + (4+x^2)^{1/2}]$

$f'(x) = \frac{1 + \frac{1}{\cancel{2}}(4+x^2)^{-1/2}(\cancel{2}x)}{x+\sqrt{4+x^2}}$

$f'(x) = \frac{1 + \frac{x}{\sqrt{4+x^2}}}{x+\sqrt{4+x^2}}$

$f'(x) = 1 + \frac{x}{\sqrt{4+x^2}} \cdot \frac{1}{x+\sqrt{4+x^2}}$

$\boxed{f'(x) = \left(1 + \frac{x}{\sqrt{4+x^2}}\right)\left(\frac{1}{x+\sqrt{4+x^2}}\right)}$

Finding a Derivative Implicitly In Exercises 73–76, use implicit differentiation to find dy/dx.

73. $x^2 - 3\ln y + y^2 = 10$

$$2x - 3\left(\frac{1}{y}\right)\frac{dy}{dx} + 2y\left(\frac{dy}{dx}\right) = 0$$

$$2x - \frac{3}{y}\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$

$$2y\left(\frac{dy}{dx}\right) - \frac{3}{y}\left(\frac{dy}{dx}\right) = -2x$$

$$\frac{dy}{dx}\left(2y - \frac{3}{y}\right) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y - \frac{3}{y}} \cdot \frac{y}{y} = \boxed{\frac{-2xy}{2y^2 - 3}}$$

74. $\ln xy + 5x = 30$

*expand first

$$\ln(xy) + 5x = 30$$

$$\ln x + \ln y + 5x = 30$$

$$\frac{1}{x} + \frac{1}{y}\left(\frac{dy}{dx}\right) + 5 = 0$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = -5 - \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{-5 - \frac{1}{x}}{\frac{1}{y}} \cdot \frac{xy}{xy} = \boxed{\frac{-5xy - y}{x}}$$

75. $4x^3 + \ln y^2 + 2y = 2x$

*expand first:

$$4x^3 + 2\ln y + 2y = 2x$$

$$12x^2 + 2\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) + 2\left(\frac{dy}{dx}\right) = 2$$

$$\frac{2}{y}\left(\frac{dy}{dx}\right) + 2\left(\frac{dy}{dx}\right) = 2 - 12x^2$$

$$\frac{dy}{dx}\left(\frac{2}{y} + 2\right) = 2 - 12x^2$$

$$\frac{dy}{dx} = \frac{2 - 12x^2}{\frac{2}{y} + 2} \cdot \frac{y}{y} = \frac{2y - 12x^2y}{2 + 2y}$$

$$= \frac{2(y - 6x^2y)}{2(1+y)}$$

$$= \boxed{\frac{y - 6x^2y}{1+y}}$$

76. $4xy + \ln x^2y = 7$

*expand first:

$$4xy + \ln(x^2y) = 7$$

$$4xy + \ln x^2 + \ln y = 7$$

$$\underbrace{4xy}_{\text{*product rule:}} + 2\ln x + \ln y = 7$$

$$(4)(y) + (4x)\left(\frac{dy}{dx}\right) + 2\left(\frac{1}{x}\right) + \frac{1}{y}\left(\frac{dy}{dx}\right) = 0$$

$$4x\left(\frac{dy}{dx}\right) + \frac{1}{y}\left(\frac{dy}{dx}\right) = -4y - \frac{2}{x}$$

$$\frac{dy}{dx}\left(4x + \frac{1}{y}\right) = -4y - \frac{2}{x}$$

$$\frac{dy}{dx} = \frac{-4y - \frac{2}{x}}{4x + \frac{1}{y}} \cdot \frac{xy}{xy} = \boxed{\frac{-4xy^2 - 2y}{4x^2y + x}}$$

Log Differentiation:
Use log differentiation to find dy/dx

1) Take "ln" of both sides
2) Expand using log properties
3) Take derivative (implicit diff. on left side)

Log Properties

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

89. $y = x\sqrt{x^2 + 1}, \quad x > 0$

$$\ln y = \ln\left[x\sqrt{x^2+1}\right]$$
$$\ln y = \ln x + \ln\sqrt{x^2+1}$$
$$\ln y = \ln x + \ln(x^2+1)^{1/2}$$
$$\ln y = \ln x + \frac{1}{2}\ln(x^2+1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1}$$
$$\frac{dy}{dx} = y\left[\frac{1}{x} + \frac{x}{x^2+1}\right]$$
$$\boxed{\frac{dy}{dx} = \left[x\sqrt{x^2+1}\right]\left[\frac{1}{x} + \frac{x}{x^2+1}\right]}$$

90. $y = \sqrt{x^2(x+1)(x+2)}, \quad x > 0$

$$\ln y = \ln\left(x^2(x+1)(x+2)\right)^{1/2}$$
$$\ln y = \frac{1}{2}\ln\left[x^2(x+1)(x+2)\right]$$
$$\ln y = \frac{1}{2}\ln x^2 + \frac{1}{2}\ln(x+1) + \frac{1}{2}(x+2)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{2}\cdot\frac{2x}{x^2} + \frac{1}{2}\cdot\frac{1}{x+1} + \frac{1}{2}\cdot\frac{1}{x+2}$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{2(x+2)}$$
$$\frac{dy}{dx} = y\left[\frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{2(x+2)}\right] = \boxed{\sqrt{x^2(x+1)(x+2)}\left[\frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{2(x+2)}\right]}$$

91. $y = \dfrac{x^2\sqrt{3x-2}}{(x+1)^2}, \quad x > \dfrac{2}{3}$

$$\ln y = \ln\left[\frac{x^2(3x-2)^{1/2}}{(x+1)^2}\right]$$
$$\ln y = \ln x^2 + \ln(3x-2)^{1/2} - \ln(x+1)^2$$
$$\ln y = 2\ln x + \frac{1}{2}\ln(3x-2) - 2\ln(x+1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = 2\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{3}{3x-2} - 2\cdot\frac{1}{x+1}$$
$$\frac{dy}{dx} = y\left[\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x+1}\right] = \boxed{\frac{x^2\sqrt{3x-2}}{(x+1)^2}\left[\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x+1}\right]}$$

92. $y = \sqrt{\dfrac{x^2-1}{x^2+1}}, \quad x > 1$

$$\ln y = \ln\left(\frac{x^2-1}{x^2+1}\right)^{1/2}$$
$$\ln y = \frac{1}{2}\ln\left(\frac{x^2-1}{x^2+1}\right)$$
$$\ln y = \frac{1}{2}\ln(x^2-1) - \frac{1}{2}\ln(x^2+1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{2}\cdot\frac{2x}{x^2-1} - \frac{1}{2}\cdot\frac{2x}{x^2+1}$$
$$\frac{dy}{dx} = y\cdot\left[\frac{x}{x^2-1} - \frac{x}{x^2+1}\right]$$
$$\boxed{\frac{dy}{dx} = \sqrt{\frac{x^2-1}{x^2+1}}\left[\frac{x}{x^2-1} - \frac{x}{x^2+1}\right]}$$

93. $y = \dfrac{x(x-1)^{3/2}}{\sqrt{x+1}}, \quad x > 1$

$$\ln y = \ln\left[\frac{x(x-1)^{3/2}}{(x+1)^{1/2}}\right]$$
$$\ln y = \ln x + \ln(x-1)^{3/2} - \ln(x+1)^{1/2}$$
$$\ln y = \ln x + \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{3}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+1}$$
$$\frac{dy}{dx} = y\left[\frac{1}{x} + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}\right]$$
$$\boxed{\frac{dy}{dx} = \frac{x(x-1)^{3/2}}{\sqrt{x+1}}\left[\frac{1}{x} + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}\right]}$$