5.2 Natural Log Integrals (Notes)

Recall Derivative Rule:

$$\frac{d}{dx}\ln u = \frac{1}{u}\cdot u'$$

Natural Log Integral Rule:

$$\int \frac{1}{u}\,du = \ln|u| + C$$

Power Rule would fail:

$$\int \frac{1}{u}\,du = \int u^{-1}\,du = \cancel{\frac{u^0}{0}}$$

$\boxed{\text{Ex. 1}}$ $\displaystyle\int \frac{2x}{x^2+1}\,dx$

$\boxed{\text{Ex. 2}}$ $\displaystyle\int \frac{1}{x\ln x}\,dx$

$\boxed{\text{Ex. 3}}$ $\displaystyle\int \frac{x^2+x+1}{x^2+1}\,dx$

5.2 (continued)

More Trig Integral Rules:

1) $\int \tan u\,du = -\ln|\cos u| + C$

2) $\int \cot u\,du = -\ln|\sin u| + C$

3) $\int \sec u\,du = \ln|\sec u + \tan u| + C$

4) $\int \csc u\,du = -\ln|\csc u + \cot u| + C$

Deriving Rules using U-substitution:

$\boxed{Ex\,4}$ a) $\int \tan x\,dx =$

b) $\int \cot x\,dx =$

$\boxed{Ex.\,5}$ Long Division to rewrite Integrand

$$\int \frac{x^3 - 6x - 20}{x+5}\,dx$$

Calculus **Ch. 5.4 Notes** **Integrals of e^x**

Integral Exponential Rule (base e): $\int e^u \, du = e^u + C$

Recall: $\frac{d}{dx} e^u = e^u \times u'$

Ex. 1: Find $\int e^{3x+1} dx$

Ex. 2: Find $\int \cos x \cdot e^{\sin x} \, dx$

Ex. 3: Find $\int \frac{e^x}{2+e^x} dx$

Ex. 4: Find $\int e^x \cos(e^x)dx$

Ex. 5: Find $\int \frac{e^{2x}}{e^{2x}+1}dx$

Ex. 6: Find $\int \frac{2e^x - 2e^{-x}}{e^x + e^{-x}}dx$

Recall Rules: $\frac{d}{dx}e^u =$ $\int e^u\,du =$

$\frac{d}{dx}a^u =$ $\int a^u\,du =$

*Remember: ln a is a **constant**

Ex. 1: $\int 2^x\,dx =$

Ex. 2: $\int 3^{4x}\,dx =$

Ex. 3: $\int 5^{\tan x}\sec^2 x\,dx =$

Ex. 4: $\int (3-x)7^{(3-x)^2}\,dx$

Ex. 5: $\int \frac{2\left(7^{2x^2-5}\right)}{5-7^{2x^2-5}}\,x\,dx$

Ex. 6: $\int_6^{-12} 4^{\frac{x}{3}}\,dx$

5.2-5.5 Mixed Review Worksheet

1. $$\int \frac{6x+5}{3x^2+5x-2}\,dx$$

2. $$\int \frac{\cos 3x}{5+2\sin 3x}\,dx$$

3. $$\int (2t+1)e^{5t^2+5t}\,dt$$

4. $$\int \frac{12x+10}{9x^2+15x-6}\,dx$$

5. $$\int \frac{e}{x^2}\cot\left(\frac{7}{x}\right)dx$$

6. Evaluate $$\int_e^{e^4} \frac{5}{x\sqrt{\ln x}}\,dx$$

7. $$\int_0^1 \frac{1+e^{3x}}{e^{3x}+3x}\,dx$$

8. $$\int \frac{\ln^3 3x}{3x}\,dx$$

9. $$\int \frac{2x^3+5x^2-12}{x+3}\,dx$$

5.2 Natural Log Integrals (Notes)

Recall Derivative Rule:

$$\frac{d}{dx}\ln u = \frac{1}{u}\cdot u'$$

Natural Log Integral Rule:

$$\int \frac{1}{u}\,du = \ln|u| + C$$

Power Rule would fail:

$$\int \frac{1}{u}\,du = \int u^{-1}\,du = \cancel{\frac{u^0}{0}}$$

Ex. 1 $\displaystyle\int \frac{2x}{x^2+1}\,dx$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$$\int \frac{\cancel{2x}}{u}\cdot\frac{du}{\cancel{2x}}$$

$$\int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|x^2+1| + C}$$

Ex. 2 $\displaystyle\int \frac{1}{x\ln x}\,dx$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

$$\int \frac{1}{\cancel{x}\cdot u}\cdot \cancel{x}\,du$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \boxed{\ln|\ln x| + C}$$

Ex 3 $\displaystyle\int \frac{x^2+x+1}{x^2+1}\,dx$

$$= \int \frac{x + x^2 + 1}{x^2+1}\,dx$$

$$= \int \frac{x}{x^2+1} + \frac{x^2+1}{x^2+1}\,dx$$

$$= \int \frac{x}{x^2+1}\,dx + \int 1\,dx$$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$$\int \frac{\cancel{x}}{u}\cdot\frac{du}{2\cancel{x}}$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u|$$

x

$$= \boxed{\frac{1}{2}\ln|x^2+1| + x + C}$$

5.2 (continued)

More Trig Integral Rules:

1) $\int \tan u\,du = -\ln|\cos u| + c$ 3) $\int \sec u\,du = \ln|\sec u + \tan u| + c$

2) $\int \cot u\,du = -\ln|\sin u| + c$ 4) $\int \csc u\,du = -\ln|\csc u + \cot u| + C$

Deriving Rules using U-substitution:

$\boxed{\text{Ex.3}}$ a) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$u = \cos x$, $\frac{du}{dx} = -\sin x$, $dx = \frac{du}{-\sin x}$

$\int \frac{\cancel{\sin x}}{u}\cdot\frac{du}{-\cancel{\sin x}} = -\int \frac{1}{u}\,du$

$= -\ln|u| + c = \boxed{-\ln|\cos x| + C}$

b) $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$

$u = \sin x$, $\frac{du}{dx} = \cos x$, $dx = \frac{du}{\cos x}$

$\int \frac{\cancel{\cos x}}{u}\cdot\frac{du}{\cancel{\cos x}}$

$= \int \frac{1}{u}\,du = \ln|u| + c = \boxed{\ln|\sin x| + C}$

$\boxed{\text{Ex.4}}$ Long Division to rewrite Integrand

$\int \frac{x^3 - 6x - 20}{x+5}\,dx$

$$
\begin{array}{r|l}
 & x^2 - 5x + 19 + \frac{-115}{x+5} \\
x+5 & x^3 - 6x - 20 \\
 & -(x^3 + 5x^2) \\
\hline
 & -5x^2 - 6x - 20 \\
 & -(-5x^2 - 25x) \\
\hline
 & 19x - 20 \\
 & -(19x + 95) \\
\hline
 & -115
\end{array}
$$

$\int x^2 - 5x + 19 - \frac{115}{x+5}\,dx$

$\int \frac{115}{x+5}\,dx$ $u = x+5$, $\frac{du}{dx} = 1$, $dx = du$

$\int \frac{115}{u}\,du = 115\ln|u| + c$

$= 115\ln|x+5| + c$

$= \boxed{\frac{x^3}{3} - \frac{5x^2}{2} + 19x - 115\ln|x+5| + C}$

5.4 Notes Integrals of e^x

Exponential Rule (Base): $\boxed{\int e^u\,du = e^u + C}$

Recall: $\frac{d}{dx}e^u = e^u \cdot u'$

Ex. 1 $\int e^{3x+1}\,dx$

$u = 3x+1$ $\quad dx = \frac{du}{3}$

$\frac{du}{dx} = 3$

$\int e^u \cdot \frac{du}{3}$

$\frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \boxed{\frac{1}{3}e^{3x+1} + C}$

2) $\int \cos x \cdot e^{\sin x}\,dx$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$\int \cancel{\cos x} \cdot e^u \cdot \frac{du}{\cancel{\cos x}}$

$\int e^u\,du$

$= e^u + C$

$= \boxed{e^{\sin x} + C}$

3) $\int \frac{e^x}{2+e^x}\,dx$

$u = 2+e^x$

$\frac{du}{dx} = e^x$

$dx = \frac{du}{e^x}$

$\int \frac{\cancel{e^x}}{u} \cdot \frac{du}{\cancel{e^x}}$

$\int \frac{1}{u}\,du$

$= \ln|u| + C = \boxed{\ln|2+e^x| + C}$

4) $\int e^x \cos(e^x)\,dx$

$u = e^x$ $\quad dx = \frac{du}{e^x}$

$\frac{du}{dx} = e^x$

$\int \cancel{e^x} \cos u \cdot \frac{du}{\cancel{e^x}}$

$\int \cos u\,du = \sin u + c = \boxed{\sin(e^x) + C}$

5) $\int \frac{e^{2x}}{1+e^{2x}}\,dx$

$u = 1 + e^{2x}$

$\frac{du}{dx} = e^{2x} \cdot 2$

$dx = \frac{du}{2e^{2x}}$

$\int \frac{\cancel{e^{2x}}}{u} \cdot \frac{du}{2\cancel{e^{2x}}}$

$\frac{1}{2}\int \frac{1}{u}\,du$

$= \frac{1}{2}\ln|u| + c$

$= \boxed{\frac{1}{2}\ln|1+e^{2x}| + C}$

6) $\int \frac{2e^x - 2e^{-x}}{e^x + e^{-x}}\,dx$

$u = e^x + e^{-x}$

$\frac{du}{dx} = e^x + e^{-x} \cdot (-1)$

$\frac{du}{dx} = e^x - e^{-x}$

$dx = \frac{du}{e^x - e^{-x}}$

$\int \frac{2\cancel{(e^x - e^{-x})}}{u} \cdot \frac{du}{\cancel{e^x - e^{-x}}}$

$2\int \frac{1}{u}\,du$

$= 2\ln|u| + c = \boxed{2\ln|e^x + e^{-x}| + C}$

Key

Recall Rules: $\frac{d}{dx}e^u = e^u \cdot u'$ $\qquad \int e^u du = e^u + C$

$\frac{d}{dx}a^u = \ln a \cdot a^u \cdot u'$ $\qquad \int a^u du = \frac{1}{\ln a}a^u + C$

*Remember: ln a is a **constant**

Ex. 1: $\int 2^x dx = \boxed{\frac{1}{\ln 2}\cdot 2^x + C \text{ or } \frac{2^x}{\ln 2} + C}$

$a = 2$

Ex. 2: $\int 3^{4x} dx =$

$u = 4x$

$\frac{du}{dx} = 4$

$4dx = du$

$dx = \frac{du}{4}$

$= \int 3^u \cdot \frac{du}{4}$

$= \frac{1}{4}\int 3^u du$

$\frac{1}{4}\cdot\frac{1}{\ln 3}\cdot 3^u + C$

$\boxed{\frac{1}{4\ln 3}\cdot 3^{4x} + C}$

Ex. 3: $\int 5^{\tan x}\sec^2 x\, dx =$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$\sec^2 x\, dx = du$

$dx = \frac{du}{\sec^2 x}$

$\int 5^u \cdot \cancel{\sec^2 x}\cdot\frac{du}{\cancel{\sec^2 x}}$

$\int 5^u du = \frac{1}{\ln 5}\cdot 5^u + C$

$= \boxed{\frac{1}{\ln 5}\cdot 5^{\tan x} + C}$

Ex. 4: $\int (3-x)7^{(3-x)^2}\, dx$

$u = (3-x)^2$

$\frac{du}{dx} = 2(3-x)^1(-1)$

$-2(3-x)dx = du$

$dx = \frac{du}{-2(3-x)}$

$\int \cancel{(3-x)}\cdot 7^u \cdot \frac{du}{-2\cancel{(3-x)}}$

$-\frac{1}{2}\int 7^u du = -\frac{1}{2}\cdot\frac{1}{\ln 7}7^u + C$

$= \boxed{\frac{-1}{2\ln 7}\cdot 7^{(3-x)^2} + C}$

Ex. 5: $\int \frac{2\left(7^{2x^2-5}\right)}{5-7^{2x^2-5}}\, x\, dx$

$*\frac{d}{dx}a^u = \ln a \cdot a^u \cdot u'$

$u = 5 - 7^{2x^2-5}$

$\frac{du}{dx} = -\ln 7 \cdot 7^{2x^2-5}\cdot(4x)$

$dx = \frac{du}{(-\ln 7)(4x)\cdot 7^{2x^2-5}}$

$\int \frac{2\cdot \cancel{7^{2x^2-5}}}{u}\cancel{x}\cdot\frac{du}{(-\ln 7)(4\cancel{x})\cancel{7^{2x^2-5}}}$

$\frac{-2}{\ln 7(4)}\int \frac{1}{u}du$

$\frac{-1}{2\ln 7}\ln|u| + C$

$= \boxed{\frac{-1}{2\ln 7}\ln\left|5-7^{2x^2-5}\right| + C}$

Ex. 6: $\int_6^{-12} 4^{\frac{x}{3}}\, dx$

$u = \frac{1}{3}x \qquad \frac{du}{dx} = \frac{1}{3} \qquad dx = 3du$

$\int 4^u \cdot 3du = 3\int 4^u du = 3\cdot\frac{1}{\ln 4}\cdot 4^u \Big]_2^{-4}$

Convert bounds:
if $x = 6$, $u = \frac{1}{3}(6) = 2$
if $x = -12$, $u = \frac{1}{3}(-12) = -4$

$= \frac{3}{\ln 4}(4^{-4}) - \frac{3}{\ln 4}(4^2)$

$= \boxed{\frac{3}{(4^4)\ln 4} - \frac{48}{\ln 4}}$

5.2-5.5 Mixed Review Worksheet

Key

1.

$$\int \frac{6x+5}{3x^2+5x-2}dx$$

$u=3x^2+5x-2$

$\frac{du}{dx}=6x+5$

$dx=\frac{du}{6x+5}$

$\int \frac{\cancel{6x+5}}{u}\cdot\frac{du}{\cancel{6x+5}}$

$\int \frac{1}{u}du=\ln|u|+C$

$=\boxed{\ln|3x^2+5x-2|+C}$

2.

$$\int \frac{\cos 3x}{5+2\sin 3x}dx$$

$u=5+2\sin(3x)$

$\frac{du}{dx}=2\cos(3x)\cdot 3$

$dx=\frac{du}{6\cos(3x)}$

$\int \frac{\cancel{\cos(3x)}}{u}\cdot\frac{du}{6\cancel{\cos(3x)}}$

$=\frac{1}{6}\int\frac{1}{u}du$

$=\frac{1}{6}\ln|u|+C$

$=\boxed{\frac{1}{6}\ln|5+2\sin(3x)|+C}$

3.

$$\int (2t+1)e^{5t^2+5t}dt$$

$u=5t^2+5t$

$\frac{du}{dt}=10t+5 \quad dt=\frac{du}{10t+5}$

$\int(2t+1)\cdot e^u\cdot\frac{du}{10t+5}$

$\int\cancel{(2t+1)}e^u\cdot\frac{du}{5\cancel{(2t+1)}}$

$\frac{1}{5}\int e^u du=\frac{1}{5}e^u+C=\boxed{\frac{1}{5}e^{5t^2+5t}+C}$

4.

$$\int \frac{12x+10}{9x^2+15x-6}dx$$

$u=9x^2+15x-6$

$\frac{du}{dx}=18x+15$

$dx=\frac{du}{18x+15}$

$\int\frac{12x+10}{u}\cdot\frac{du}{18x+15}$

$\int\frac{2\cancel{(6x+5)}}{u}\cdot\frac{du}{3\cancel{(6x+5)}}$

$=\frac{2}{3}\int\frac{1}{u}du=\frac{2}{3}\ln|u|+C$

$=\boxed{\frac{2}{3}\ln|9x^2+15x-6|+C}$

5.

$$\int \frac{2^{\sin x}}{3+2^{\sin x}}\cos x\,dx$$

$u=3+2^{\sin x}$

$\frac{du}{dx}=\ln 2\cdot 2^{\sin x}\cdot\cos x$

$dx=\frac{du}{\ln 2\cdot 2^{\sin x}\cdot\cos x}$

$\int\frac{\cancel{2^{\sin x}}}{u}\cancel{\cos x}\cdot\frac{du}{\ln 2\cdot\cancel{2^{\sin x}}\cdot\cancel{\cos x}}$

$\frac{1}{\ln 2}\int\frac{1}{u}du$

$\frac{1}{\ln 2}\ln|u|+C$

$\boxed{\frac{1}{\ln 2}\ln|3+2^{\sin x}|+C}$

6.

Evaluate $\int_e^{e^4}\frac{5}{x\sqrt{\ln x}}dx$ $\quad\int\frac{5}{x(\ln x)^{1/2}}dx$

$u=\ln x \quad \frac{du}{dx}=\frac{1}{x} \quad dx=x\,du$

$\int\frac{5}{\cancel{x}\cdot u^{1/2}}\cancel{x}du=5\int\frac{1}{u^{1/2}}du$

$5\int u^{-1/2}du=\frac{5u^{1/2}}{1/2}=10u^{1/2}\Big]_1^4$

if $x=e$, $u=\ln e=1$

if $x=e^4$, $u=\ln e^4=4$

$=10(4)^{1/2}-10(1)^{1/2}$

$=20-10=\boxed{10}$

7.

$$\int_0^1\frac{1+e^{3x}}{e^{3x}+3x}dx$$

$\int\frac{1+e^{3x}}{u}\cdot\frac{du}{3e^{3x}+3}$

$u=e^{3x}+3x$

$\frac{du}{dx}=e^{3x}(3)+3$

$dx=\frac{du}{3e^{3x}+3}$

$\int\frac{\cancel{1+e^{3x}}}{u}\cdot\frac{du}{3\cancel{(e^{3x}+1)}}$

$\frac{1}{3}\int\frac{1}{u}du=$

$\frac{1}{3}\ln|u|\Big]_1^{e^3+3}$

if $x=0$, $u=1$

if $x=1$, $u=e^3+3$

$\frac{1}{3}\ln|e^3+3|-0$

$=\boxed{\frac{1}{3}\ln|e^3+3|}$

8.

$$\int\frac{\ln^3 3x}{3x}dx=\int\frac{[\ln(3x)]^3}{3x}dx$$

$u=\ln(3x)$

$\frac{du}{dx}=\frac{3}{3x}=\frac{1}{x}$

$dx=x\,du$

$\int\frac{u^3}{3\cancel{x}}\cdot\cancel{x}du$

$\frac{1}{3}\int u^3du$

$=\frac{1}{3}\left(\frac{u^4}{4}\right)+C$

$=\boxed{\frac{1}{12}(\ln(3x))^4+C}$

9.

$$\int\frac{2x^3+5x^2-12}{x+3}dx$$

$$\begin{array}{r|l}
 & 2x^2-x+3-\frac{21}{x+3} \\
x+3 & 2x^3+5x^2-12 \\
 & \ominus 2x^3+6x^2 \\
 & -x^2-12 \\
 & +x^2+3x \\
 & 3x-12 \\
 & \ominus 3x+9 \\
 & -21
\end{array}$$

OR synthetic division

$$\begin{array}{r|rrrr}
-3 & 2 & 5 & 0 & -12 \\
 & \downarrow & -6 & 3 & -9 \\
\hline
 & 2 & -1 & 3 & -21
\end{array}$$

$2x^2-x+3-\frac{21}{x+3}$

$u=x+3 \quad \frac{du}{dx}=1$

$21\int\frac{1}{u}du=21\ln|u|$

$\int 2x^2-x+3-\frac{21}{x+3}dx$

$\boxed{\frac{2x^3}{3}-\frac{x^2}{2}+3x-21\ln|x+3|+C}$