5.2 Exercise Problems – Mean Value Theorem (MVT) and Rolle's Theorem

p. 327–331 #9, 17, 25, 27, 31, 37, 68

Rolle's Theorem:
i) $f(x)$ is continuous $[a,b]$
ii) $f(x)$ is differentiable (a,b)
iii) $f(a) = f(b)$

$f'(c) = 0$ for some c-value in interval (a,b)

Apply Rolle's Theorem:

9) $f(x) = x^3 - x \quad [-1,0]$

i) $f(x)$ continuous $[-1,0]$
ii) $f(x)$ differentiable $(-1,0)$
iii) $f(-1) = 0$, $f(0) = 0$, so $f(-1) = f(0)$

$f'(x) = 3x^2 - 1$
$0 = 3x^2 - 1$

$3x^2 - 1 = 0$
$3x^2 = 1$

$x^2 = 1/3$
$x = \pm\sqrt{1/3}$

$\cancel{c = \frac{\sqrt{3}}{3}}$, $c = -\frac{\sqrt{3}}{3}$

outside interval

$\boxed{c = \frac{-\sqrt{3}}{3}}$

17) $f(x) = x^2 - 2x + 1$ on $[-2,1]$

i) $f(x)$ is continuous $[-2,1]$
ii) $f(x)$ is differentiable $(-2,1)$
iii) $f(-2) = 9$, $f(1) = 0$

Since $f(-2) \neq f(1)$, Rolle's Theorem cannot be applied on interval $[-2,1]$

5.2 Mean Value Theorem (MVT)

Apply Mean Value Theorem

Mean Value Theorem (MVT)
Conditions:

i) $f(x)$ is continuous $[a,b]$
ii) $f(x)$ is differentiable (a,b)

Then $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

for some c-value on the interval (a,b)

25) $f(x) = x^3 - 5x^2 + 4x - 2$ on $[1,3]$

i) $f(x)$ is continuous $[1,3]$
ii) $f(x)$ is differentiable $(1,3)$

$f(1) = -2$ | slope: $\dfrac{-8-^-2}{3-1} \to \dfrac{-6}{2} = -3$
$f(3) = -8$

set equal to each other

$f'(x) = 3x^2 - 10x + 4$

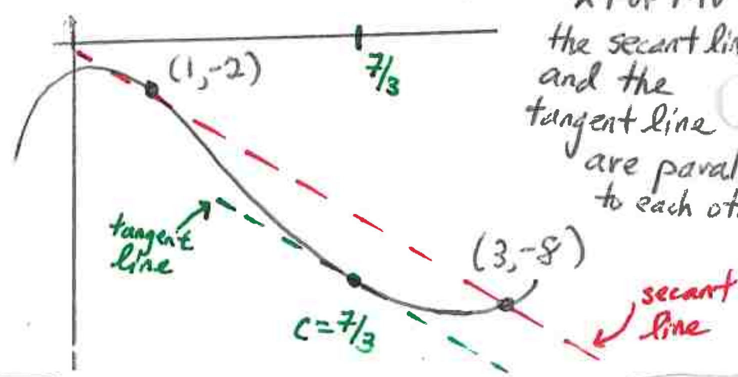
$-3 = 3x^2 - 10x + 4$

$0 = 3x^2 - 10x + 7$

$0 = (3x-7)(x-1)$

$3x - 7 = 0$ | $x - 1 = 0$
$x = 7/3$ | $\cancel{x = 1}$ outside interval

$\boxed{c = 7/3 \text{ on interval } (1,3)}$



*For MVT, the secant line and the tangent line are parallel to each other.

27) $f(x) = \dfrac{x+1}{x}$ on $[1,3]$ **Apply MVT**

i) $f(x)$ continuous $[1,3]$ (* There is vertical asymptote at $x=0$, but this discontinuity does not interfere with interfere with interval $(1,3)$)
ii) $f(x)$ is differentiable $(1,3)$

$f(1) = \frac{2}{1} = 2$, $f(3) = \frac{4}{3}$

slope: $\dfrac{4/3 - 2}{3-1} \to \dfrac{-2/3}{2} \to \dfrac{-1}{3}$

$f'(x) = \dfrac{\overbrace{(1)}^{f'}\overbrace{(x)}^{g} - \overbrace{(x+1)}^{f}\overbrace{(1)}^{g'}}{\underbrace{x^2}_{g^2}}$

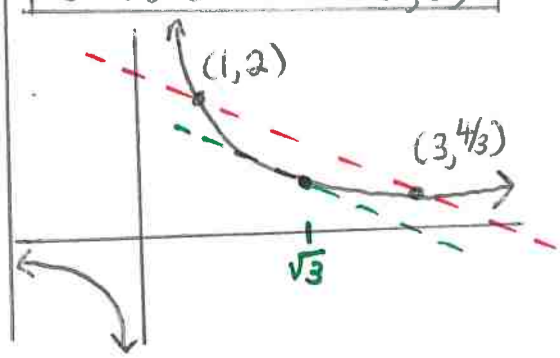
set equal

$f'(x) = \dfrac{x-x-1}{x^2} \to f'(x) = \dfrac{-1}{x^2}$

$\dfrac{-1}{x^2} = \dfrac{-1}{3} \to x^2 = 3 \to$ $x = \pm\sqrt{3}$

$c = \sqrt{3}$, $\cancel{c = -\sqrt{3}}$ outside interval

$\boxed{c = \sqrt{3} \text{ on interval } (1,3)}$



5.2 Determine Intervals where f(x) is increasing and decreasing

31) $f(x) = x^3 + 6x^2 + 12x + 1$

*set $f'(x)=0$ to find critical points.

$$f'(x) = 3x^2 + 12x + 12$$

$$0 = 3x^2 + 12x + 12$$

$$0 = 3(x^2 + 4x + 4)$$

$$0 = 3(x+2)(x+2)$$

$$x = -2$$

$f'(x)$: test points -3: $+$ $\nearrow$; critical point -2; test point 0: $+$ $\nearrow$

f(x) is increasing $(-\infty, -2), (-2, \infty)$ since $f'(x) > 0$

f(x) is never decreasing on any interval.

37) $f(x) = x^{2/3}(x^2 - 4)$

$$f'(x) = \underbrace{\tfrac{2}{3}x^{-1/3}}_{f'}\underbrace{(x^2-4)}_{g} + \underbrace{x^{2/3}}_{f} \cdot \underbrace{2x}_{g'}$$

$$f'(x) = \frac{2(x^2-4)}{3x^{1/3}} + \frac{2x(x^{2/3})}{1}$$

$$f'(x) = \frac{2x^2-8}{3x^{1/3}} + \frac{2x \cdot x^{2/3} \cdot 3x^{1/3}}{3x^{1/3}}$$

$$f'(x) = \frac{2x^2 - 8 + 6x^2}{3x^{1/3}} \rightarrow \frac{8x^2-8}{3x^{1/3}}$$

$$f'(x) = \frac{8(x^2-1)}{3x^{1/3}} \rightarrow \boxed{f'(x) = \frac{8(x+1)(x-1)}{3x^{1/3}}}$$

*set numerator $=0$ and denominator $=0$ to find critical points:

$8(x^2-1) = 0$	$3x^{1/3} = 0$
$8(x-1)(x+1) = 0$	$x^{1/3} = 0$
$x = 1, x = -1$	$x = 0$

$f'(x)$: test point -2: $-$ $\searrow$; critical point -1; test point $-\tfrac{1}{2}$: $+$ $\nearrow$; critical point 0; test point $\tfrac{1}{2}$: $-$ $\searrow$; critical point 1; test point 2: $+$ $\nearrow$

f(x) is increasing on $(-1, 0), (1, \infty)$ since $f'(x) > 0$

f(x) is decreasing on $(-\infty, -1), (0, 1)$ since $f'(x) < 0$

$\boxed{5.2}$

68) Let $f(x) = x^{2/3}$ on interval $[-1, 1]$

There is no c in $(-1, 1)$ for which $\frac{f(1) - f(-1)}{1 - (-1)}$

Explain why this does not contradict Mean Value Theorem.

The conditions for MVT do not pass for this function on this interval

✓ i) f(x) is continuous on $[-1, 1]$

X ii) f(x) is not differentiable on $(-1, 1)$

$f'(x) = \frac{2}{3}x^{-1/3}$

$f'(x) = \frac{2}{3x^{1/3}}$

set denominator = 0

$3x^{1/3} = 0$

$x^{1/3} = 0$

$\boxed{x = 0}$

* Any critical point coming from the denominator of $f'(x)$ signifies a value where the graph is not differentiable (this is where the derivative is undefined and therefore slope does not exist, and that means graph is not smooth, or not differentiable at that point)

f(x) is not differentiable at $x = 0$. Since this value interferes with the interval $(-1, 1)$, f(x) is not differentiable on $(-1, 1)$ and do not pass the required conditions of MVT to guarantee a slope value.