AP Calculus – 5.2 Notes – Mean Value Theorem (MVT) and Rolle's Theorem

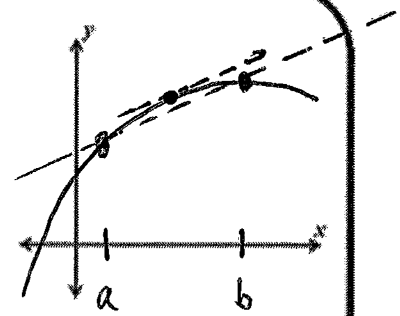
Key

We use the MVT to justify conclusions about a function over an interval.

Mean Value Theorem:

If a function f is continuous over the interval $[a,b]$ and differentiable over the interval (a,b) then there exists a point c within that open interval where the instantaneous rate of change equals the average rate of change over the interval.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

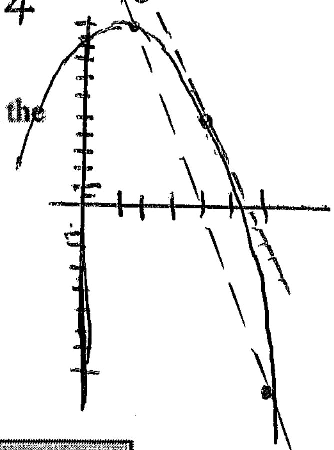


1. Use the function $f(x) = -x^2 + 3x + 10$ to answer the following.
   a. On the interval [2, 6], what is the average rate of change? slope between endpts

$f(2) = -2^2 + 3(2) + 10 = 12$

$f(6) = -6^2 + 3(6) + 10 = -8$

$\frac{f(6) - f(2)}{6 - 2} \rightarrow \frac{-8 - 12}{4} \rightarrow \frac{-20}{4} = -5$



   b. On the interval (2, 6), when does the instantaneous rate of change equal the average rate of change?

$f'(x) = -2x + 3$

$-2x + 3 = -5$

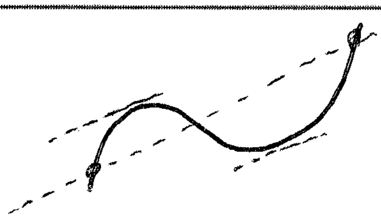
$-2x = -8$

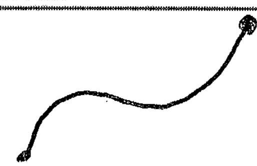
$x = 4$

$c = 4$

MVT vs IVT

Mean Value Theorem MVT	Intermediate Value Theorem IVT
• The derivative (instantaneous rate of change) must equal the average rate of change somewhere in the interval.	• On a given interval, you will have a y-value at each of the end points of the interval. Every y-value exists between these two y-values at least once in the interval.





2.

t minutes	0	5	15	20	30
$h(t)$ feet	0	40	70	65	80

A hot air balloon is launched into the air with a human pilot. The twice-differentiable function h models the balloon's height, measured in feet, at time t, measured in minutes. The table above gives values of the $h(t)$ of the balloon at selected times t.

* differentiable implies continuity

a. For $5 \le t \le 15$, must there be a time t when the balloon is 50 feet in the air? Justify your answer.

$h(t)$ is continuous on $[5,15]$ (By IVT)

Since $h(5) = 40 < 50 < 70 = h(15)$, there must be a time t on $[5,15]$ where $h(t) = 50$

b. For $20 \le t \le 30$, must there be a time t when the balloon's velocity is 1.5 feet per minute? Justify your answer.

* Find Avg. Rate of Change

$h(t)$ is continuous $[20,30]$

$h(t)$ differentiable $(20,30)$

$\dfrac{h(30)-h(20)}{30} = \dfrac{80-65}{10} = \dfrac{15}{10} = 1.5$ ft/min

By MVT, there must be a value where $h'(t) = 1.5$ on $(20,30)$

Practice:

Skater Sully is riding a skateboard back and forth on a street that runs north/south. The twice-differentiable function S models Sully's position on the street, measured by how many meters north he is from his starting point, at time t, measured in seconds from the start of his ride. The table below gives values of $S(t)$ at selected times t.

t seconds	0	20	30	60
$S(t)$ meters	0	−5	7	40

a. For $0 \le t \le 20$, must there be a time t when Sully is 2 meters south of his starting point? Justify your answer.

$S(t)$ is continuous on $[0,20]$

By IVT, since $S(20) = -5 < -2 < 0 = S(0)$, there must be a value c such that $S(c) = -2$ on $[0,20]$

b. For $30 \le t \le 60$, must there be a time t when Sully's velocity is 1.1 meters per second? Justify your answer.

$S(t)$ continuous on $[30,60]$

$S(t)$ differentiable on $(30,60)$

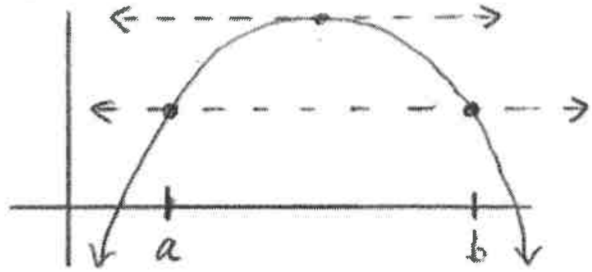
$\dfrac{S(60)-S(30)}{60-30} = \dfrac{40-7}{60-30} = 1.1$

By MVT, there must be a value where $S'(c) = 1.1$ on $(30,60)$

Rolle's Theorem: If a function, f(x) is continuous on [a, b], differentiable on (a, b), and f(a) = f(b), then there must be at least one point on the function where the slope of the tangent (derivative) is 0.

*In other words, if the endpoints have the same y-values, then we can guarantee a relative maximum or relative minimum somewhere between the endpoints

*Rolle's Theorem is just a specific case of the Mean Value Theorem



$$\frac{f(b)-f(a)}{b-a} = 0$$

Rolle's Theorem Steps:

1. Check Continuity (no breaks between endpoints)
2. Check Differentiability (smooth curve between endpoints)

Note, all polynomials are continuous and differentiable everywhere

3. Test endpoints. Does f(a) = f(b)? If not ,then Rolle's fails / does not apply
4. If yes, then set f'(x) = 0 and solve for x

Example 2: Determine if Rolle's theorem can be applied to $f(x) = x^2 - 3x + 2$ on the interval [1, 2]. If so, find the value of c such that f '(c) = 0.

f(x) is continuous [1, 2]
f(x) is differentiable (1, 2)
$f(1) = 0$
$f(2) = 0$

slope $= m = \frac{0-0}{2-1} = 0$
$f'(x) = 2x - 3$

$2x - 3 = 0$
$x = 3/2$
$\boxed{c = 3/2}$

Using the Mean Value Theorem, find where the instantaneous rate of change is equivalent to the average rate of change.

4. $y = x^2 - 5x + 2$ on $[-4, -2]$

f(x) continuous [-4, -2], differentiable (-4, -2)
$f(-4) = (-4)^2 - 5(-4) + 2 = 38$
$f(-2) = (-2)^2 - 5(-2) + 2 = 16$
$m = \frac{38-26}{-2-(-4)} = \frac{12}{2} = \boxed{6}$
$y' = \boxed{2x-5}$

$2x - 5 = 6$
$2x = 11$
$x = 11/2$ $\boxed{c = 11/2}$

5. $y = \sin 3x$ on $[0, \pi]$

f(x) continuous [-4, -2], differentiable (-4, -2)
$f(0) = \sin 0 = 0$
$f(\pi) = \sin 3\pi = 0$
$m = \frac{0-0}{\pi - 0} = 0$
$y' = \cos(3x) \cdot 3$
$3\cos(3x) = 0$
$\cos(3x) = 0$
$3x = \cos^{-1}(0)$

$\left(3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}\right) \div 3$
$\boxed{x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \cancel{\frac{7\pi}{6}}}$

$f(x)$ continuous $[1,3]$, differentiable $(1,3)$ $f(x)$ continuous $[0, \ln 2]$, differentiable $(0, \ln 2)$

6. $y = (-5x + 15)^{\frac{1}{2}}$ on $[1, 3]$

$y(1) = 10^{1/2} \rightarrow \frac{0 - \sqrt{10}}{3 - 1} = \frac{-\sqrt{10}}{2}$

$y(3) = 0^{1/2}$

$y' = \frac{1}{2}(-5x+15)^{-1/2}(-5)$

$y' = \frac{-5}{2(-5x+15)^{1/2}}$

$\frac{-5}{2\sqrt{-5x+15}} = \frac{-\sqrt{10}}{2}$

$(-5)^2 = \left(\sqrt{10}\left(\sqrt{-5x+15}\right)\right)^2$

$25 = 10(-5x+15)$ | $25 = -50x + 150$

$50x = 125$

$\boxed{x = 2.5}$

7. $y = e^x$ on $[0, \ln 2]$

$y(\ln 2) = e^{\ln 2} = 2$

$y(0) = e^0 = 1$

$m = \frac{2-1}{\ln 2 - 0} = \frac{1}{\ln 2}$

$y' = e^x$

$\frac{1}{\ln 2} = e^x$

$e^x = (\ln 2)^{-1}$

$\ln e^x = \ln(\ln 2)^{-1}$

$\boxed{x = \ln\left(\frac{1}{\ln 2}\right)}$

$x = \ln(\ln 2)^{-1}$

Calculator active problem

8. A particle moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = t^3 - 3t^2 + t + 1$. For what values of t, $0 \leq t \leq 2$, is the particle's instantaneous velocity the same as its average velocity on the closed interval $[0, 2]$?

$x(0) = 1$

$x(2) = 8 - 12 + 2 + 1 = -1$

$m = \frac{-1-1}{2-0} = \frac{-2}{2} = -1$

$v(t) = 3t^2 - 6t + 1$

$3t^2 - 6t + 1 = -1$

$3t^2 - 6t - 2 = 0$

$\boxed{t \approx 0.4226,\ 1.577}$

No calculator on this problem.

9. The table below gives selected values of a function f. The function is twice differentiable with $f''(x) > 0$.

this means $f'(x)$ is increasing

x	$f(x)$
3	12.5
5	13.9
7	16.1

$\frac{f(5) - f(3)}{5-3} \rightarrow \frac{1.4}{2} = 0.7$

$\frac{f(7) - f(5)}{7-5} \rightarrow \frac{2.2}{2} \rightarrow 1.1$

0.7

1.1

Which of the following could be the value of $f'(5)$?

(A) 0.5 (B) 0.7 $\boxed{\text{(C) } 0.9}$ (D) 1.1 (E) 1.3

$f'(5)$ must be between 0.7 and 1.1

11.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	8	2	4
2	6	3	1	2
3	5	−3	6	3
4	−2	6	3	5

The functions f and g are differentiable for all real numbers. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) + 2$. Must there be a value c for $2 < c < 4$ such that $h'(c) = 1$.

$h(4) = f(g(4)) + 2 = f(3) + 2 \rightarrow 5 + 2 = 7$

$h(2) = f(g(2)) + 2 = f(1) + 2 = 3 + 2 = 5$

$\frac{h(4) - h(2)}{4-2} \rightarrow \frac{7-5}{2} = 1$

By MVT, since the avg. rate of change is 1, there must be instantaneous rate of change of 1 on $(2,4)$ interval.