5.3 Exercise Problems 1st Derivative Test, Test for Concavity, and 2nd Derivative Test

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p. 343–347 #13, 19, 29, 41, 47, 49, 51, 59, 67, 69, 77, 92, 94, 99

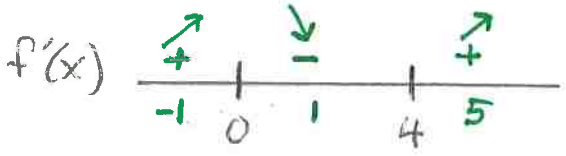
Find critical values. Use 1st Derivative Test to find relative extrema

13) $f(x) = x^3 - 6x^2 + 2$

$f'(x) = 3x^2 - 12x$

$0 = 3x(x-4)$

$x = 0,\ x = 4$



f(x) is increasing $(-\infty, 0), (4, \infty)$ b/c $f'(x) > 0$

f(x) is decreasing $(0, 4)$ b/c $f'(x) < 0$

Relative max at point $(0, \underline{2})$ b/c $f'(x)$ changes from + to − at $x = 0$

Relative min at point $(4, \underline{-30})$ b/c $f'(x)$ changes from − to + at $x = 4$

19) $f(x) = x^{2/3} + x^{1/3}$

$f'(x) = \frac{2}{3}x^{-1/3} + \frac{1}{3}x^{-2/3}$

$f'(x) = \frac{2}{3x^{1/3}} + \frac{1}{3x^{2/3}}$

$= \frac{x^{1/3} \cdot 2}{3x^{2/3}} + \frac{1}{3x^{2/3}}$

$f'(x) = \frac{2x^{1/3} + 1}{3x^{2/3}}$

$2x^{1/3} + 1 = 0$

$2x^{1/3} = -1$

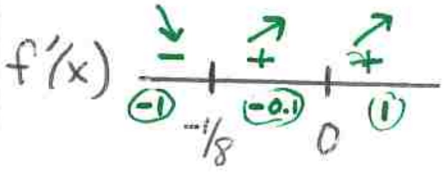
$x^{1/3} = -\frac{1}{2}$

$\underline{\underline{x = -1/8}}$

$3x^{2/3} = 0$

$x^{2/3} = 0$

$\underline{\underline{x = 0}}$



f(x) is increasing $(-1/8, 0), (0, \infty)$ b/c $f'(x) > 0$

f(x) is decreasing $(-\infty, -1/8)$ b/c $f'(x) < 0$

Relative minimum at $(-1/8, \underline{-1/4})$ b/c $f'(x)$ changes from − to + at $x = -1/8$

* set numerator = 0 and denominator = 0 separately to find critical pts.

$\boxed{5.3}$ particle motion problem (along the x-axis)

29) $s(t) = 2t^3 + 6t^2 - 18t + 1$ for $t \geq 0$

a) Determine interval where particle moves left, moves right

$v(t) = s'(t) = 6t^2 + 12t - 18$

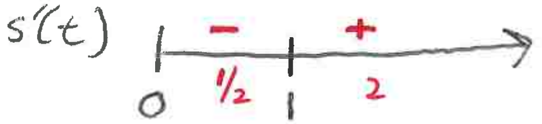
$0 = 6(t^2 + 2t - 3)$

$0 = 6(t+3)(t-1)$

$t+3=0$ | $t-1=0$

$\cancel{t=-3}$ | $t=1$

outside interval



Particle moves to the right on the interval $(1, \infty)$ since $s'(t) > 0$

Particle moves left on the interval $[0, 1)$ since $s'(t) < 0$

b) When does object reverse direction?

particle reverses direction at $t=1$ since $s'(t)$ changes signs.

c) When is the velocity of the object increasing? when is velocity decreasing?

*To find intervals of velocity increasing / velocity decreasing, find critical pts and sign line for acceleration $a(t)$ $(s''(t))$

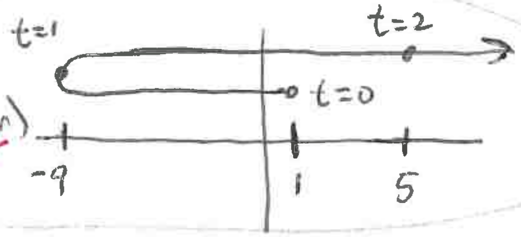
$a(t) = 12t + 12$ | $\cancel{t=-1}$

$0 = 12(t+1)$ | outside interval

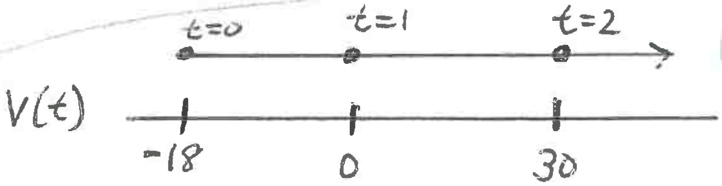
$t+1=0$

Since $a(t)$ is always positive, velocity is increasing on interval $[0, \infty)$

d) figure illustrating motion of object → (direction)



e) Figure illustrating the velocity of the object



$\boxed{5.3}$ Find Local (Relative Extrema) and Use Test for Concavity (POI)

41) $f(x) = x^4 - 4x$

$f'(x) = 4x^3 - 4$

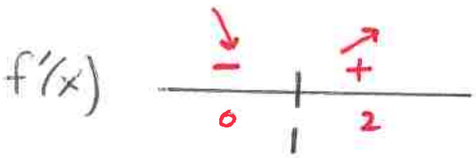
diff. of cubes $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$0 = 4(x^3 - 1)$

$0 = 4(x-1)(x^2+1x+1)$

$x - 1 = 0$

$\underline{\underline{x = 1}}$

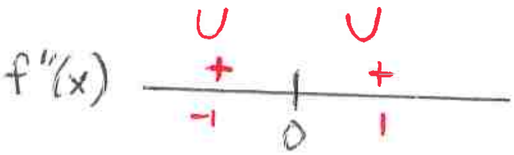


a) Relative minimum at point $(1, \underline{-3})$ since f'(x) changes from − to +

$f''(x) = 12x^2$

$0 = 12x^2$

$x = 0$



b) f(x) is concave up $(-\infty, 0), (0, \infty)$ b/c $f''(x) > 0$

c) No point of inflection on f(x)

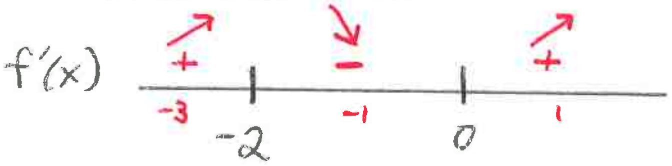
47) $f(x) = x^2 e^x$

$f'(x) = \overbrace{2x}^{f'} \cdot \overbrace{e^x}^{g} + \overbrace{x^2}^{f} \cdot \overbrace{e^x}^{g'}$

$f'(x) = xe^x(2 + x)$

$0 = (x)(e^x)(2 + x)$

$\underline{\underline{x = 0}}$ | $e^x \neq 0$ | $2 + x = 0$, $\underline{\underline{x = -2}}$



a) Relative max at point $(-2, \underline{4/e^2})$ since f'(x) changes from + to −

Relative min at point $(0, 0)$ since f'(x) changes from − to +

$f'(x) = e^x(2x + x^2)$

$f''(x) = \overbrace{e^x}^{f'} \overbrace{(2x+x^2)}^{g} + \overbrace{e^x}^{f} \overbrace{(2+2x)}^{g'}$

$f''(x) = e^x(x^2 + 4x + 2)$

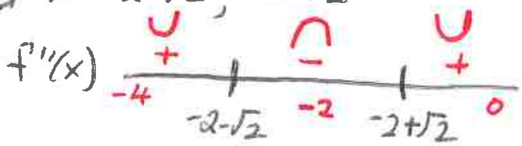
$e^x \neq 0$ | $x^2 + 4x + 2 = 0$

$$\frac{-4 \pm \sqrt{4^2 - (4 \cdot 1 \cdot 2)}}{2(1)}$$

$$\frac{-4 \pm \sqrt{8}}{2} \rightarrow \frac{-4}{2} \pm \frac{2\sqrt{2}}{2} \rightarrow -2 \pm \sqrt{2}$$

$x = -2 - \sqrt{2}, -2 + \sqrt{2}$

b)



c) POI at $x = -2 - \sqrt{2}$ and $x = -2 + \sqrt{2}$ since f''(x) change signs.

$\boxed{5.3}$ Find Relative Extrema and Use Test for Concavity (Find POI)

49) $f(x) = 6x^{4/3} - 3x^{1/3}$

$$f'(x) = 6\cdot\tfrac{4}{3}x^{1/3} - 3\cdot\tfrac{1}{3}x^{-2/3}$$

$$f'(x) = 8x^{1/3} - \frac{1}{x^{2/3}}$$

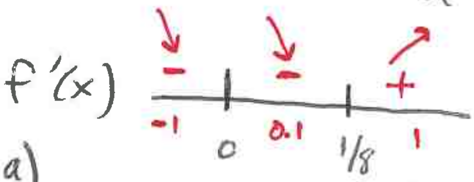
$$f'(x) = \frac{x^{2/3}\cdot 8x^{1/3}}{x^{2/3}} - \frac{1}{x^{2/3}}$$

$$f'(x) = \frac{8x-1}{x^{2/3}}$$

∗ find critical points: $8x - 1 = 0$ | $x^{2/3} = 0$

$8x = 1$ | $x = 0$

$x = 1/8$



a)
Relative minimum at $(1/8, \underline{-9/8})$ because $f'(x)$ changes from − to +

$$f''(x) = \frac{\overbrace{8}^{f'}(\overbrace{x^{2/3}}^{g}) \overset{-}{-} (\overbrace{8x-1}^{f})\overbrace{\tfrac{2}{3}x^{-1/3}}^{g'}}{\underbrace{(x^{2/3})^2}_{g^2}} \rightarrow \frac{8x^{2/3} - \frac{16}{3}x^{2/3} + \frac{2}{3x^{1/3}}}{x^{4/3}}$$

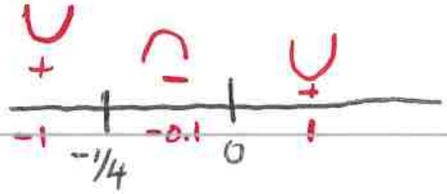
$$f''(x) = \frac{\frac{8}{3}x^{2/3} + \frac{2}{3x^{1/3}}}{x^{4/3}} \rightarrow \frac{\frac{x^{1/3}\cdot 8x^{2/3}}{3x^{1/3}} + \frac{2}{3x^{1/3}}}{x^{4/3}} \rightarrow \frac{\frac{8x+2}{3x^{1/3}}}{x^{4/3}}$$

$$f''(x) = \frac{8x+2}{3x^{5/3}}$$

→ ∗ critical pts → $8x + 2 = 0$ | $3x^{5/3} = 0$

$x = -1/4$ | $x = 0$

b) $f''(x)$



c) Point of Inflection at $\left(-1/4, \underline{\frac{9(\sqrt[3]{2})}{4}}\right)$ and $(0, \underline{0})$ since $f''(x)$ change signs.

5.3

51) $f(x) = x^{2/3}(x^2-8)$

$f(x) = x^{8/3} - 8x^{2/3}$

$f'(x) = \frac{8}{3}x^{5/3} - 8\cdot\frac{2}{3}x^{-1/3}$

$f'(x) = \frac{8x^{5/3}}{3} - \frac{16}{3x^{1/3}}$

$f'(x) = \frac{x^{1/3}\cdot 8x^{5/3}}{3x^{1/3}} - \frac{16}{3x^{1/3}}$

$f'(x) = \frac{8x^2-16}{3x^{1/3}}$

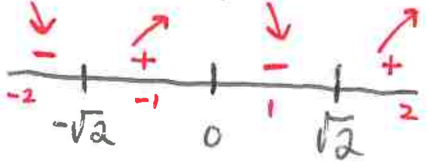
* find critical points:

$8x^2-16=0$ | $3x^{1/3}=0$

$x^2=2$ | $x=0$

$x=\pm\sqrt{2}$

a) $f'(x)$



Relative maximum at $(0, \underline{0})$ because $f'(x)$ changes from + to −

Relative minimums at $(-\sqrt{2}, \underline{-6\sqrt[3]{2}})$ and $(\sqrt{2}, \underline{-6\sqrt[3]{2}})$ b/c $f'(x)$ changes from − to +.

$f''(x) = \frac{8}{3}\cdot\frac{5}{3}x^{2/3} - \frac{16}{3}\cdot\frac{-1}{3}x^{-4/3}$

$f''(x) = \frac{40}{3}x^{2/3} + \frac{16}{3x^{4/3}}$

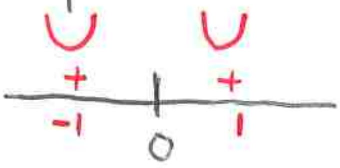
$f''(x) = \frac{x^{4/3}\cdot 40x^{2/3}}{3x^{4/3}} + \frac{16}{3x^{4/3}}$

$f''(x) = \frac{40x^2+16}{3x^{4/3}}$

$40x^2+16=0$ | $3x^{4/3}=0$

no critical pts. | $x=0$

b) $f''(x)$



c) There is no Point of Inflection since $f''(x)$ does not change signs.

$\boxed{5.3}$

59) $f(x) = x - 2\sin x \qquad 0 \le x \le 2\pi$

$f'(x) = 1 - 2\cos x$

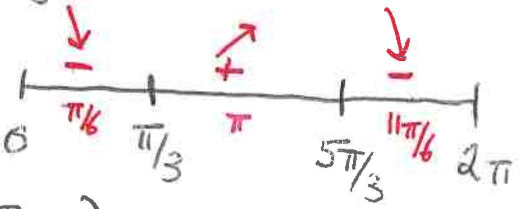
$0 = 1 - 2\cos x$

$2\cos x = 1$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

$f'(x)$



a) Relative minimum at $\left(\frac{\pi}{3}, \underline{\frac{\pi}{3}-\sqrt{3}}\right)$ because $f'(x)$ changes from − to +

Relative maximum at $\left(\frac{5\pi}{3}, \underline{\frac{5\pi}{3}+\sqrt{3}}\right)$ b/c $f'(x)$ changes from + to −

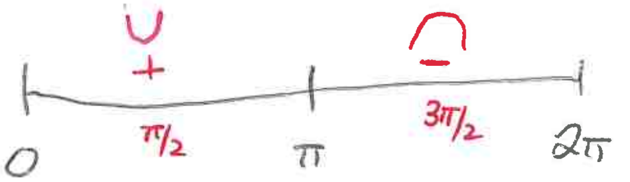
$f''(x) = -2(-\sin x)$

$f''(x) = 2\sin x$

$0 = 2\sin x$

$x = 0, \pi, 2\pi$

b) $f''(x)$



$f(x)$ is concave up on $(0, \pi)$ since $f''(x) > 0$

$f(x)$ is concave down on $(\pi, 2\pi)$ since $f''(x) < 0$

c) Point of Inflection at $(\pi, \underline{\pi})$ since $f''(x)$ change signs.

$\boxed{5.3}$ Use First Derivative Test and 2nd Derivative Test to find Local Extrema (Relative Max/Min)

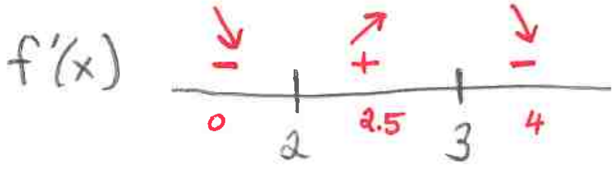
67) $f(x) = -2x^3 + 15x^2 - 36x + 7$

$f'(x) = -6x^2 + 30x - 36$

$0 = -6(x^2 - 5x + 6)$

$0 = -6(x-3)(x-2)$

$x = 3,\ x = 2$



Relative minimum at $(2, \underline{-21})$ since $f'(x)$ changes from − to +

Relative maximum at $(3, \underline{-20})$ since $f'(x)$ changes from + to −

<u>2nd Derivative Test (Steps)</u>

i) Identify critical point candidates from $f'(x)$

ii) Find $f''(x)$

iii) Insert $f'(x)$ candidate critical values into $f''(x)$

- a) If $f''(a) < 0$, then Relative Maximum occurs at $x = a$
- b) If $f''(b) > 0$, then Relative Minimum occurs at $x = b$
- c) If $f''(c) = 0$, then Relative Extrema is Inconclusive (Use 1st Derivative Test)

$f''(x) = -12x + 30$

$f''(2) = -12(2) + 30 = 6 > 0$ Relative minimum occurs at $x = 2$

$f''(3) = -12(3) + 30 = -6 < 0$ Relative maximum occurs at $x = 3$

$\boxed{5.3}$ Apply First and 2nd Derivative Test to find Local Extrema

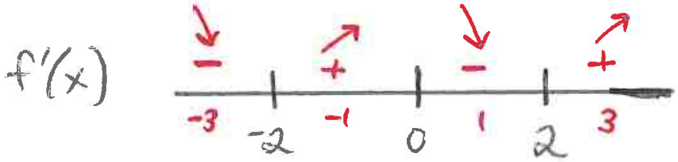
69) $f(x) = x^4 - 8x^2 - 5$

$f'(x) = 4x^3 - 16x$

$0 = 4x(x^2 - 4)$

$0 = 4x(x+2)(x-2)$

$x=0,\ x=-2,\ x=2$



Relative minimum at points $(-2, \underline{-21})$ and $(2, \underline{-21})$ b/c $f'(x)$ changes from − to +

Relative maximum at $(0, \underline{-5})$ since $f'(x)$ changes from + to −

*Apply 2nd Derivative Test

$f''(x) = 12x^2 - 16$ ← *test candidates $x = -2, 0, 2$

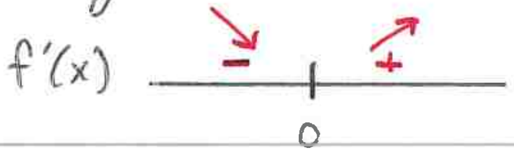
$f''(-2) = 12(-2)^2 - 16 = 32 > 0$ Relative minimum at $x=-2$

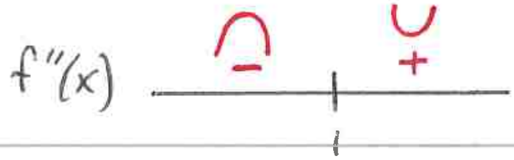
$f''(0) = 12(0)^2 - 16 = -16 < 0$ Relative maximum at $x=0$

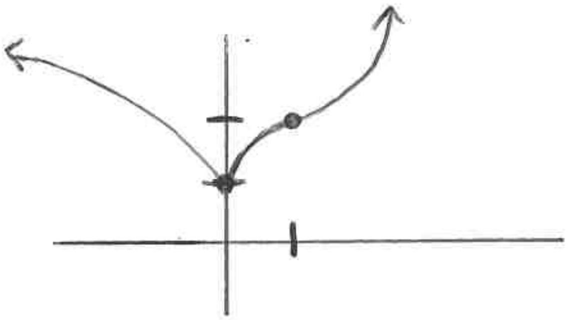
$f''(2) = 12(2)^2 - 16 = 32 > 0$ Relative minimum at $x=2$

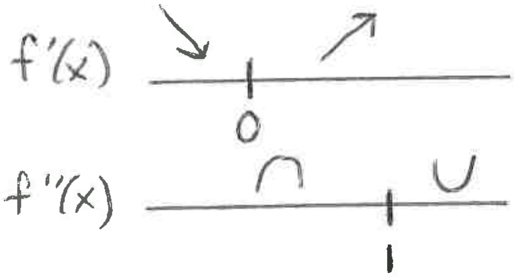
Sketch Graph with the given properties:

77) f is concave down $(-\infty, 1)$, concave up on $(1, \infty)$, decreasing on $(-\infty, 0)$, increasing on $(0, \infty)$, $f(0)=1$, $f(1)=2$









$\boxed{5.3}$

92) Inflection Point: For the function $f(x) = ax^3 + bx^2 + cx + d$, find a, b, c, and d so that 0 is a critical number, $f(0) = 4$ and the point $(1, -2)$ is an inflection point of f.

* since $f(0) = 4$, $4 = a(0)^3 + b(0)^2 + c(0) + d$, so $\underline{\underline{d = 4}}$

* $f'(x) = 3ax^2 + 2bx + c$ $f'(0) = 0$ since $x = 0$ is a critical point.

$f'(0) = 3a(0)^2 + 2b(0) + c$

$0 = 0 + 0 + c \rightarrow \underline{\underline{c = 0}}$

* $f''(x) = 6ax + 2b$ Since $x = 1$ is an inflection point, $f''(1) = 0$

$f''(1) = 6a(1) + 2b$

$\boxed{0 = 6a + 2b}$

* plug in point $(1, -2)$ into $f(x)$ equation

$f(x) = ax^3 + bx^2 + cx + d$

$f(1) = a(1)^3 + b(1)^2 + c(1) + d$

$-2 = a + b + c + d$ ← simplify equation since we know $c = 0$, $d = 4$

$-2 = a + b + 0 + 4$

$\boxed{-6 = a + b}$

* Solve for a and b since $6a + 2b = 0$ and $a + b = -6$

$b = -a - 6 \rightarrow$

$6a + 2(-a - 6) = 0$

$6a - 2a - 12 = 0$

$4a - 12 = 0$

$4a = 12$

$\underline{\underline{a = 3}}$

$a + b = -6$

$3 + b = -6$

$\underline{\underline{b = -9}}$

$\boxed{a = 3,\ b = -9,\ c = 0,\ d = 4}$

$\boxed{5.3}$

94) Profit Function $P(t) = \frac{300}{1+50e^{-0.2t}}$ where $t \geq 0$

a) When is annual profit increasing? When is profit decreasing?
* determine when $P'(t) > 0$ or $P'(t) < 0$

$$P(t) = 300(1+50e^{-0.2t})^{-1}$$

$$P'(t) = -300(1+50e^{-0.2t})^{-2}(50e^{-0.2t} \cdot -0.2)$$

$$\boxed{P'(t) = \frac{3000e^{-0.2t}}{(1+50e^{-0.2t})^2}}$$

*Since $P'(t) > 0$ for all t-values, profit is always increasing $(0, \infty)$ and never decreasing.

b) ⟶ Find the rate of change in profit $P'(t)$

c) When is rate of change of profit increasing? decreasing?
* The wording here is referencing $P''(t)$

$$P''(t) = \frac{3000e^{-0.2t} \cdot (-0.2)(1+50e^{-0.2t})^2 - 3000e^{-0.2t} \cdot 2(1+50e^{-0.2t})(50e^{-0.2t})(-0.2)}{(1+50e^{-0.2t})^4}$$

$$P''(t) = \frac{600e^{-0.2t}(-1-50e^{-0.2t}+100e^{-0.2t})}{(1+50e^{-0.2t})^3} \rightarrow \frac{600e^{-0.2t}(50e^{-0.2t}-1)}{(1+50e^{-0.2t})^3}$$

*set $P''(t) = 0$

$$50e^{-0.2t} - 1 = 0$$

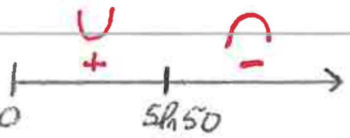
$$e^{-0.2t} = \frac{1}{50}$$

$$\ln e^{-0.2t} = \ln\left(\frac{1}{50}\right)$$

$$-0.2t \cancel{\ln e} = \ln(1/50)$$

$$t = 5\ln(50)$$

$P''(t)$



$P'(t)$ is increasing $(0, 5\ln 50)$ b/c $P''(t) > 0$

$P'(t)$ is decreasing $(5\ln 50, \infty)$ b/c $P''(t) < 0$

d) Rate of change at maximum when $t = 5\ln 50$

e) Inflection Point of $P(t)$ is $(5\ln 50, \underline{150})$

f) Inflection Point is where $P'(t)$ is at a maximum

$\boxed{5.3}$

99) If $f(x)=ax^3+bx^2+cx+d$, $a\neq 0$, find a, b, c, and d.

Local minimum at $x=0$. Local max at $x=4$

Points on graph are $(0,5)$ and $(4,33)$

* At local min and max, the slope $=0$, so $f'(x)=0$ at $x=0$, $x=4$

$f'(x)=3ax^2+2bx+c$

At $x=0$

$f'(0)=3a(0)^2+2b(0)+c$

$0=c$

At $x=4$

$f'(4)=3a(4)^2+2b(4)+c$

$0=48a+8b+0$

$0=48a+8b$

* Plug in points $(0,5)$ and $(4,33)$ to gather more info on missing constants

$f(x)=ax^3+bx^2+cx+d$

$f(0)=a(0)^3+b(0)^2+c(0)+d$

$5=d$

$f(4)=a(4)^3+b(4)^2+c(4)+d$

$33=64a+16b+4c+d$

c=0 d=5

$33=64a+16b+0+5$

$28=64a+16b$

$48a+8b=0$

$64a+16b=28$

$-2(48a+8b=0)$

$64a+16b=28$

$\rightarrow$

$-96a-\cancel{16b}=0$

$64a+\cancel{16b}=28$

$-32a=28$

$a=\frac{-28}{32}=\frac{-7}{8}$ $\quad a=\frac{-7}{8}$

$48a+8b=0$

$48\left(\frac{-7}{8}\right)+8b=0$

$-42+8b=0$

$8b=42$

$b=\frac{42}{8}\rightarrow\frac{21}{4}$

$\boxed{a=\frac{-7}{8},\ b=\frac{21}{4},\ c=0,\ d=5}$