and

$$\lim_{x\to 0^+} A(x) = 0, \quad \text{and} \quad \lim_{x\to\infty} A(x) = \lim_{x\to\infty} \frac{2x}{e^{x^2}} = \lim_{x\to\infty} \frac{1}{xe^{x^2}} = 0,$$

A has an absolute maximum when $x = \frac{\sqrt{2}}{2}$. Observe that this is the x-coordinate of one of the points of inflection of y, so the rectangle of largest area that can be inscribed under the graph of $y = e^{-x^2}$ has two of its vertices at the points of inflection of y.

53. Because distance is non-negative, minimizing the square of the distance will produce the same result as minimizing the distance but does not require the use of square roots. Let D denote the square of the distance between an arbitrary point (x, y) on the graph of $y = e^{-x/2}$ and the point $(1, 8)$. Then

$$D = (x-1)^2 + (y-8)^2 = (x-1)^2 + (e^{-x/2} - 8)^2.$$

The domain of D is all real numbers, and because D is differentiable everywhere, the critical numbers of D occur where $D'(x) = 0$. Now,

$$D'(x) = 2(x-1) + 2(e^{-x/2} - 8) \cdot \left(-\frac{1}{2}e^{-x/2}\right) = 2(x-1) - e^{-x/2}(e^{-x/2} - 8).$$

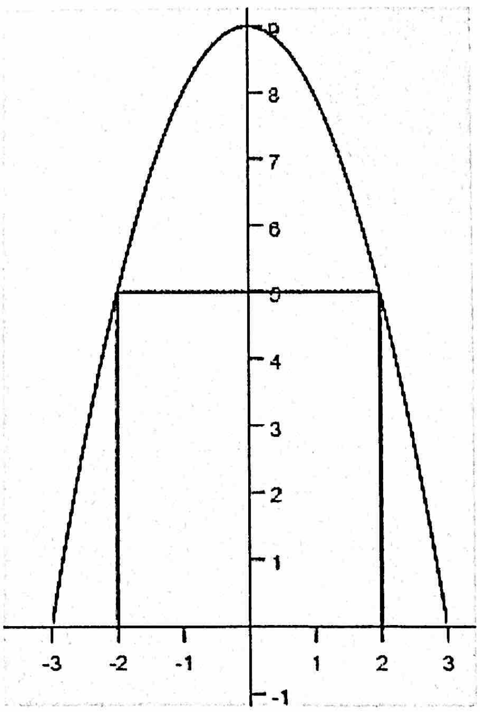
Using the computer algebra system *Maple*, the only critical number is $x \approx -3.758$. As

$$\lim_{x\to\pm\infty} D(x) = \infty,$$

it follows that D has an absolute minimum at approximately -3.758. The point $\boxed{(-3.758, e^{1.879}) \approx (-3.758, 6.547)}$ on the graph of $y = e^{-x/2}$ is closest to the point $(1, 8)$.

AP® Practice Problems

1.



Let y denote the height of the rectangle and $2x$ be the width of the base of the rectangle.

$A = wh = 2xy$ with $y = 9 - x^2$

Substituting for y in the area formula yields

$$A = 2x(9 - x^2) \\ = 18x - 2x^3$$

The domain of this function is the open interval $(-3, 3)$. The function A is differentiable on the open interval $(-3, 3)$, so the critical numbers occur where $A'(x) = 0$. Now,

$$A'(x) = 18 - 6x^2$$

Set $A'(x) = 18 - 6x^2 = 0$

$$6(3 - x^2) = 0$$

$$x = \pm\sqrt{3}$$

So the base of the rectangle is $2x = \boxed{2\sqrt{3}}$ and the height is $y = 9 - \left(\sqrt{3}\right)^2 = 9 - 3 = \boxed{6}$.

CHOICE D

2. For a closed cylindrical can the amount of aluminum used is represented by the equation $S = 2\pi rh + 2\pi r^2$ including the lateral surface area along with the top and the bottom of the cylindrical can.

$$V = \pi r^2 h = 16\pi$$

$$h = \frac{16}{r^2}$$

Substituting h into S yields

$$S = 2\pi r\left(\frac{16}{r^2}\right) + 2\pi r^2$$

$$= 32\pi r^{-1} + 2\pi r^2$$

$$S' = -32\pi r^{-2} + 4\pi r$$

Let $S' = 0$

$$4\pi r - 32\pi r^{-2} = 0$$

$$4\pi r^3 - 32\pi = 0$$

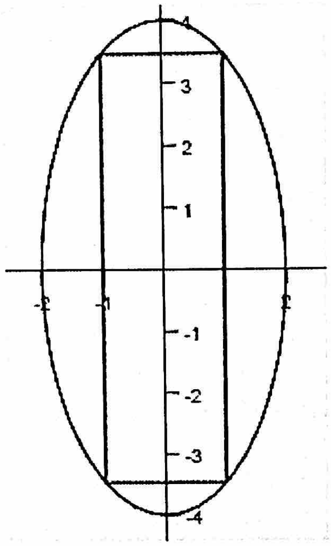
$$r^3 = 8$$

$$r = \boxed{2}$$

$$h = \frac{16}{r^2} = \frac{16}{2^2} = \boxed{4}.$$

CHOICE B

3.



For the x-coordinate and the y-coordinate on the Cartesian coordinate system, the width of the rectangle is $2x$ and the height is $2y$.

$$A = (2x)(2y) = 4xy$$

Given $4x^2 + y^2 = 16$

$$y^2 = 16 - 4x^2$$

$$y = (16 - 4x^2)^{\frac{1}{2}}$$

Substituting into A yields

$$A = 4x(16 - 4x^2)^{\frac{1}{2}}$$

$$A' = 4\left[1(16 - 4x^2)^{\frac{1}{2}} + \frac{1}{2}(16 - 4x^2)^{\frac{-1}{2}}(-8x)(x)\right]$$

$$= 4\left[(16 - 4x^2)^{\frac{1}{2}} - \frac{4x^2}{(16 - 4x^2)^{\frac{1}{2}}}\right]$$

Let $A' = 0$

$$4\left[(16 - 4x^2)^{\frac{1}{2}} - \frac{4x^2}{(16 - 4x^2)^{\frac{1}{2}}}\right] = 0$$

$$16 - 4x^2 - 4x^2 = 0$$

$$16 - 8x^2 = 0$$

$$8x^2 = 16$$

$$x^2 = 2$$

$$x = \sqrt{2}$$

$$y = \sqrt{16 - 4\left(\sqrt{2}\right)^2}$$
$$= \sqrt{16 - 8}$$
$$= \sqrt{8}$$
$$= 2\sqrt{2}$$
$$A = 4xy = 4\left(\sqrt{2}\right)\left(2\sqrt{2}\right) = \boxed{16}.$$

CHOICE D

4. Let the product $P = xy$

Given $y = 4x^2 - 3$

$P = x(4x^2 - 3) = 4x^3 - 3x$

$P' = 12x^2 - 3$

Let $P' = 12x^2 - 3 = 0$

$$4x^2 - 1 = 0$$
$$(2x - 1)(2x + 1) = 0$$
$$x = \frac{1}{2} \quad x = \frac{-1}{2}$$

	$y = 4x^2 - 3$	$P = xy$
For $x = \frac{1}{2}$	$y = 4\left(\frac{1}{2}\right)^2 - 3 = -2$	$P = \left(\frac{1}{2}\right)(-2) = -1$
For $x = -\frac{1}{2}$	$y = 4\left(-\frac{1}{2}\right)^2 - 3 = -2$	$P = \left(-\frac{1}{2}\right)(-2) = 1$

The minimum product is $\boxed{-1}$.

CHOICE A

5. The minimum distance between $(1, 0)$ and $(x - 1)y = 4$ is to be determined by using the distance formula between $(1, 0)$ and a point on $(x - 1)y = 4$.

$$D = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{(x-1)^2 + (y)^2}$$

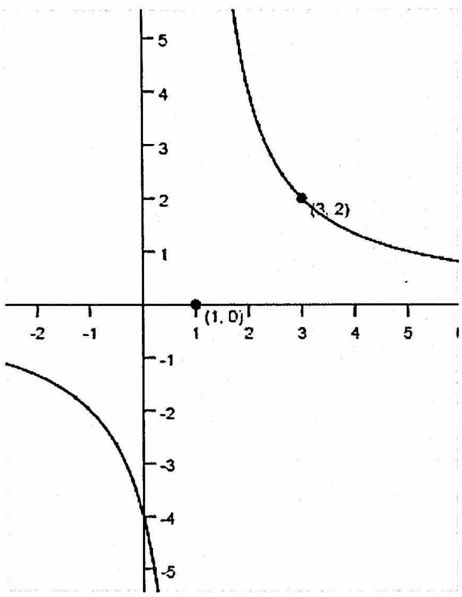
Substituting $y = \frac{4}{x-1}$ yields

$$D = \sqrt{(x-1)^2 + \left(\frac{4}{x-1}\right)^2}$$
$$D^2 = (x-1)^2 + \left(\frac{4}{x-1}\right)^2$$
$$= (x-1)^2 + 16(x-1)^{-2}$$
$$2DD' = 2(x-1) + 16(-2)(x-1)^{-3}$$

$$\text{Let } 2DD' = 0$$
$$2x - 2 - \frac{32}{(x-1)^3} = 0$$
$$2(x-1)^4 - 32 = 0$$
$$(x-1)^4 = 16$$
$$x - 1 = 2$$
$$x = 3$$
$$y = \frac{4}{x-1} = \frac{4}{3-1} = 2$$

The point on the graph of $(x-1)y = 4$ closest to $(1,0)$ is $\boxed{(3,2)}$.

CHOICE C



5.6 Antiderivatives; Differential Equations

Concepts and Vocabulary

1. A function F is called an **antiderivative** of a function f if $F' = f$.

2. True. If F is an antiderivative of f, then $F(x) + C$, where C is a constant, is also an antiderivative of f.

3. All the antiderivatives of $y = x^{-1}$ are $\boxed{\ln|x| + C, \text{ where } C \text{ is a constant}}$.

4. True. An antiderivative of $\sin x$ is $-\cos x + \pi$ because $\frac{d}{dx}(-\cos x + \pi) = \sin x$.