143. $y = \dfrac{L}{1 + ae^{-x/b}}, \quad a > 0, b > 0, L > 0$

$$y' = \frac{-L\left(-\frac{a}{b}e^{-x/b}\right)}{\left(1 + ae^{-x/b}\right)^2} = \frac{\frac{aL}{b}e^{-x/b}}{\left(1 + ae^{-x/b}\right)^2}$$

$$y'' = \frac{\left(1 + ae^{-x/b}\right)^2\left(\frac{-aL}{b^2}e^{-x/b}\right) - \left(\frac{aL}{b}e^{-x/b}\right)2\left(1 + ae^{-x/b}\right)\left(\frac{-a}{b}e^{-x/b}\right)}{\left(1 + ae^{-x/b}\right)^4}$$

$$= \frac{\left(1 + ae^{-x/b}\right)\left(\frac{-aL}{b^2}e^{-x/b}\right) + 2\left(\frac{aL}{b}e^{-x/b}\right)\left(\frac{a}{b}e^{-x/b}\right)}{\left(1 + ae^{-x/b}\right)^3} = \frac{Lae^{-x/b}\left[ae^{-x/b} - 1\right]}{\left(1 + ae^{-x/b}\right)^3 b^2}$$

$$y'' = 0 \text{ if } ae^{-x/b} = 1 \Rightarrow \frac{-x}{b} = \ln\left(\frac{1}{a}\right) \Rightarrow x = b \ln a$$

$$y(b \ln a) = \frac{L}{1 + ae^{-(b \ln a)/b}} = \frac{L}{1 + a(1/a)} = \frac{L}{2}$$

Therefore, the y-coordinate of the inflection point is $L/2$.

Section 5.5 Bases Other than e and Applications

1. $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$

2. $\log_{27} 9 = \log_{27} 27^{2/3} = \frac{2}{3}$

3. $\log_7 1 = 0$

4. $\log_a \dfrac{1}{a} = \log_a 1 - \log_a a = -1$

5. (a) $2^3 = 8$
   $\log_2 8 = 3$

   (b) $3^{-1} = \frac{1}{3}$
   $\log_3 \frac{1}{3} = -1$

6. (a) $27^{2/3} = 9$
   $\log_{27} 9 = \frac{2}{3}$

   (b) $16^{3/4} = 8$
   $\log_{16} 8 = \frac{3}{4}$

7. (a) $\log_{10} 0.01 = -2$
   $10^{-2} = 0.01$

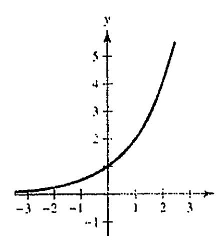
   (b) $\log_{0.5} 8 = -3$
   $0.5^{-3} = 8$
   $\left(\frac{1}{2}\right)^{-3} = 8$

8. (a) $\log_3 \frac{1}{9} = -2$
   $3^{-2} = \frac{1}{9}$

   (b) $49^{1/2} = 7$
   $\log_{49} 7 = \frac{1}{2}$

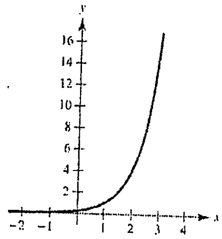
9. $y = 2^x$

x	-2	-1	0	1	2
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4



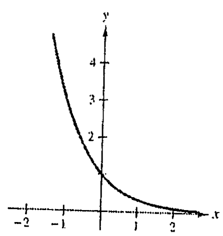
10. $y = 4^{x-1}$

x	-1	0	1	2	3
y	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16



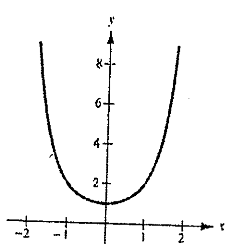
11. $y = \left(\frac{1}{3}\right)^x = 3^{-x}$

x	-2	-1	0	1	2
y	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$



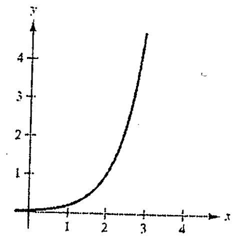
12. $y = 2^{x^2}$

x	-2	-1	0	1	2
y	16	2	1	2	16



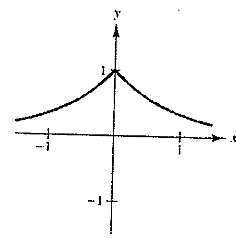
13. $h(x) = 5^{x-2}$

x	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5



14. $y = 3^{-|x|}$

x	0	± 1	± 2
y	1	$\frac{1}{3}$	$\frac{1}{9}$



15. $f(x) = 3^x$

Increasing function that passes through $(0, 1)$ and $(1, 3)$. Matches (d).

16. $f(x) = 3^{-x}$

Decreasing function that passes through $(0, 1)$ and $\left(1, \frac{1}{3}\right)$. Matches (c).

17. $f(x) = 3^x - 1$

Increasing function that passes through $(0, 0)$ and $(1, 2)$. Matches (b).

18. $f(x) = 3^{x-1}$

Increasing function that passes through $(1, 1)$ and $(2, 3)$. Matches (a).

19. (a) $\log_{10} 1000 = x$

$$10^x = 1000$$
$$x = 3$$

(b) $\log_{10} 0.1 = x$

$$10^x = 0.1$$
$$x = -1$$

20. (a) $\log_3 \frac{1}{81} = x$

$$3^x = \frac{1}{81}$$
$$x = -4$$

(b) $\log_6 36 = x$

$$6^x = 36$$
$$x = 2$$

21. (a) $\log_3 x = -1$

$$3^{-1} = x$$
$$x = \frac{1}{3}$$

(b) $\log_2 x = -4$

$$2^{-4} = x$$
$$x = \frac{1}{16}$$

22. (a) $\log_b 27 = 3$

$$b^3 = 27$$
$$b = 3$$

(b) $\log_b 125 = 3$

$$b^3 = 125$$
$$b = 5$$

23. (a)

$$x^2 - x = \log_5 25$$
$$x^2 - x = \log_5 5^2 = 2$$
$$x^2 - x - 2 = 0$$
$$(x + 1)(x - 2) = 0$$
$$x = -1 \quad \text{OR} \quad x = 2$$

(b) $3x + 5 = \log_2 64$

$$3x + 5 = \log_2 2^6 = 6$$
$$3x = 1$$
$$x = \frac{1}{3}$$

24. (a) $\log_3 x + \log_3(x - 2) = 1$

$$\log_3[x(x - 2)] = 1$$
$$x(x - 2) = 3^1$$
$$x^2 - 2x - 3 = 0$$
$$(x + 1)(x - 3) = 0$$
$$x = -1 \quad \text{OR} \quad x = 3$$

$x = 3$ is the only solution because the domain of the logarithmic function is the set of all *positive* real numbers.

(b) $\log_{10}(x + 3) - \log_{10} x = 1$

$$\log_{10} \frac{x + 3}{x} = 1$$
$$\frac{x + 3}{x} = 10^1$$
$$x + 3 = 10x$$
$$3 = 9x$$
$$x = \frac{1}{3}$$

25.

$$3^{2x} = 75$$
$$2x \ln 3 = \ln 75$$
$$x = \left(\frac{1}{2}\right)\frac{\ln 75}{\ln 3} \approx 1.965$$

26.

$$5^{6x} = 8320$$
$$6x \ln 5 = \ln 8320$$
$$x = \frac{\ln 8320}{6 \ln 5} \approx 0.935$$

27.

$$2^{3-z} = 625$$
$$(3 - z) \ln 2 = \ln 625$$
$$3 - z = \frac{\ln 625}{\ln 2}$$
$$z = 3 - \frac{\ln 625}{\ln 2} \approx -6.288$$

28.

$$3(5^{x-1}) = 86$$
$$5^{x-1} = \frac{86}{3}$$
$$(x - 1) \ln 5 = \ln\left(\frac{86}{3}\right)$$
$$x - 1 = \frac{\ln(86/3)}{\ln 5}$$
$$x = 1 + \frac{\ln(86/3)}{\ln 5} \approx 3.085$$

29. $\left(1 + \frac{0.09}{12}\right)^{12t} = 3$

$$12t \ln\left(1 + \frac{0.09}{12}\right) = \ln 3$$

$$t = \left(\frac{1}{12}\right)\frac{\ln 3}{\ln\left(1 + \frac{0.09}{12}\right)} \approx 12.253$$

30. $\left(1 + \frac{0.10}{365}\right)^{365t} = 2$

$$365t \ln\left(1 + \frac{0.10}{365}\right) = \ln 2$$

$$t = \left(\frac{1}{365}\right)\frac{\ln 2}{\ln\left(1 + \frac{0.10}{365}\right)} \approx 6.932$$

31. $\log_2(x - 1) = 5$

$$x - 1 = 2^5 = 32$$

$$x = 33$$

32. $\log_{10}(t - 3) = 2.6$

$$t - 3 = 10^{2.6}$$

$$t = 3 + 10^{2.6} \approx 401.107$$

33. $\log_3 x^2 = 4.5$

$$x^2 = 3^{4.5}$$

$$x = \pm\sqrt{3^{4.5}} \approx \pm 11.845$$

34. $\log_5 \sqrt{x - 4} = 3.2$

$$\sqrt{x - 4} = 5^{3.2}$$

$$x - 4 = \left(5^{3.2}\right)^2 = 5^{6.4}$$

$$x = 4 + 5^{6.4}$$

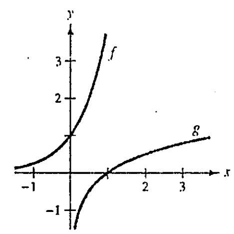
$$\approx 29{,}748.593$$

35. $f(x) = 4^x$

$g(x) = \log_4 x$

x	-2	-1	0	$\frac{1}{2}$	1
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4

x	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4
$g(x)$	-2	-1	0	$\frac{1}{2}$	1

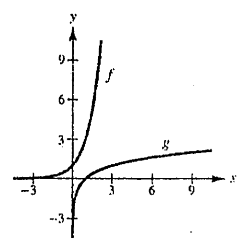


36. $f(x) = 3^x$

$g(x) = \log_3 x$

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$g(x)$	-2	-1	0	1	2



37. $f(x) = 4^x$

$$f'(x) = (\ln 4)4^x$$

38. $f(x) = 3^{4x}$

$$f'(x) = 4(\ln 3)3^{4x} = 4(\ln 3)81^x$$

39. $y = 5^{-4x}$

$$y' = -4(\ln 5)5^{-4x}$$

$$= \frac{-4 \ln 5}{625^x}$$

40. $y = 6^{3x-4}$

$y' = 3(\ln 6)6^{3x-4}$

41. $f(x) = x9^x$

$f'(x) = x(\ln 9)9^x + 9^x$

$= 9^x(1 + x\ln 9)$

42. $y = x(6^{-2x})$

$\frac{dy}{dx} = x\left[-2(\ln 6)6^{-2x}\right] + 6^{-2x}$

$= 6^{-2x}\left[-2x(\ln 6) + 1\right]$

$= 6^{-2x}(1 - 2x\ln 6)$

43. $g(t) = t^2 2^t$

$g'(t) = t^2(\ln 2)2^t + (2t)2^t$

$= t2^t(t\ln 2 + 2)$

$= 2^t t(2 + t\ln 2)$

44. $f(t) = \frac{3^{2t}}{t}$

$f'(t) = \frac{t(2\ln 3)3^{2t} - 3^{2t}}{t^2}$

$= \frac{3^{2t}(2t\ln 3 - 1)}{t^2}$

45. $h(\theta) = 2^{-\theta}\cos \pi\theta$

$h'(\theta) = 2^{-\theta}(-\pi\sin \pi\theta) - (\ln 2)2^{-\theta}\cos \pi\theta$

$= -2^{-\theta}\left[(\ln 2)\cos \pi\theta + \pi\sin \pi\theta\right]$

46. $g(\alpha) = 5^{-\alpha/2}\sin 2\alpha$

$g'(\alpha) = 5^{-\alpha/2}2\cos 2\alpha - \frac{1}{2}(\ln 5)5^{-\alpha/2}\sin 2\alpha$

47. $y = \log_4(5x + 1)$

$y' = \frac{1}{(5x+1)\ln 4}(5)$

$= \frac{5}{\ln 4(5x + 1)}$

48. $y = \log_3(x^2 - 3x)$

$y' = \frac{1}{(x^2 - 3x)\ln 3}(2x - 3)$

$= \frac{2x - 3}{x(x - 3)\ln 3}$

49. $h(t) = \log_5(4 - t)^2 = 2\log_5(4 - t)$

$h'(t) = 2\frac{-1}{\ln(5)(4 - t)} = \frac{2}{(t - 4)\ln 5}$

50. $g(t) = \log_2(t^2 + 7)^3 = 3\log_2(t^2 + 7)$

$g'(t) = 3\frac{2t}{\ln 2(t^2 + 7)} = \frac{6t}{(t^2 + 7)\ln 2}$

51. $y = \log_5\sqrt{x^2 - 1} = \frac{1}{2}\log_5(x^2 - 1)$

$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{2x}{(x^2 - 1)\ln 5} = \frac{x}{(x^2 - 1)\ln 5}$

52. $f(x) = \log_2\sqrt[3]{2x + 1} = \frac{1}{3}\log_2(2x + 1)$

$f'(x) = \frac{1}{3}\frac{1}{(2x + 1)\ln 2}(2) = \frac{2}{3(2x + 1)\ln 2}$

53. $f(x) = \log_2\frac{x^2}{x - 1} = 2\log_2 x - \log_2(x - 1)$

$f'(x) = \frac{2}{x\ln 2} - \frac{1}{(x - 1)\ln 2} = \frac{x - 2}{(\ln 2)x(x - 1)}$

54. $y = \log_{10}\frac{x^2 - 1}{x} = \log_{10}(x^2 - 1) - \log_{10} x$

$\frac{dy}{dx} = \frac{2x}{(x^2 - 1)\ln 10} - \frac{1}{x\ln 10}$

$= \frac{1}{\ln 10}\left[\frac{2x}{x^2 - 1} - \frac{1}{x}\right]$

$= \frac{1}{\ln 10}\left[\frac{x^2 + 1}{x(x^2 - 1)}\right]$

55. $h(x) = \log_3\frac{x\sqrt{x - 1}}{2}$

$= \log_3 x + \frac{1}{2}\log_3(x - 1) - \log_3 2$

$h'(x) = \frac{1}{x\ln 3} + \frac{1}{2}\cdot\frac{1}{(x - 1)\ln 3} - 0$

$= \frac{1}{\ln 3}\left[\frac{1}{x} + \frac{1}{2(x - 1)}\right]$

$= \frac{1}{\ln 3}\left[\frac{3x - 2}{2x(x - 1)}\right]$

56. $g(x) = \log_5\left(\frac{4}{x^2\sqrt{1-x}}\right)$

$= \log_5 4 - \log_5 x^2 - \log_5 \sqrt{1-x}$

$= \log_5 4 - 2\log_5 x - \frac{1}{2}\log_5(1-x)$

$g'(x) = -2\frac{1}{x \ln 5} - \frac{1}{2}\frac{1}{(1-x)\ln 5}(-1)$

$= \frac{-2}{x \ln 5} + \frac{1}{2(1-x)\ln 5}$

57. $g(t) = \frac{10 \log_4 t}{t} = \frac{10}{\ln 4}\left(\frac{\ln t}{t}\right)$

$g'(t) = \frac{10}{\ln 4}\left[\frac{t(1/t) - \ln t}{t^2}\right]$

$= \frac{10}{t^2 \ln 4}[1 - \ln t]$

$= \frac{5}{t^2 \ln 2}(1 - \ln t)$

58. $f(t) = t^{3/2}\log_2 \sqrt{t+1} = t^{3/2}\frac{1}{2}\frac{\ln(t+1)}{\ln 2}$

$f'(t) = \frac{1}{2\ln 2}\left[t^{3/2}\frac{1}{t+1} + \frac{3}{2}t^{1/2}\ln(t+1)\right]$

59. $y = 2^{-x}$, $(-1, 2)$

$y' = -2^{-x}\ln(2)$

At $(-1, 2)$, $y' = -2\ln(2)$.

Tangent line: $y - 2 = -2\ln(2)(x+1)$

$y = -2x \ln 2 + 2 - 2\ln 2$

60. $y = 5^{x-2}$, $(2, 1)$

$y' = 5^{x-2}\ln 5$

At $(2, 1)$, $y' = \ln 5$.

Tangent line: $y - 1 = \ln(5)(x - 2)$

$y = x\ln 5 + 1 - 2\ln 5$

61. $y = \log_3 x$, $(27, 3)$

$y' = \frac{1}{x \ln 3}$

At $(27, 3)$, $y' = \frac{1}{27 \ln 3}$.

Tangent line: $y - 3 = \frac{1}{27\ln 3}(x - 27)$

$y = \frac{1}{27\ln 3}x + 3 - \frac{1}{\ln 3}$

62. $y = \log_{10}(2x)$, $(5, 1)$

$y' = \frac{1}{x \ln 10}$

At $(5, 1)$, $y' = \frac{1}{5\ln 10}$.

Tangent line: $y - 1 = \frac{1}{5\ln 10}(x - 5)$

$y = \frac{1}{5\ln 10}x + 1 - \frac{1}{\ln 10}$

63. $y = x^{2/x}$

$\ln y = \frac{2}{x}\ln x$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{x}\left(\frac{1}{x}\right) + \ln x\left(-\frac{2}{x^2}\right) = \frac{2}{x^2}(1 - \ln x)$

$\frac{dy}{dx} = \frac{2y}{x^2}(1 - \ln x) = 2x^{(2/x)-2}(1 - \ln x)$

64. $y = x^{x-1}$

$\ln y = (x-1)(\ln x)$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = (x-1)\left(\frac{1}{x}\right) + \ln x$

$\frac{dy}{dx} = y\left[\frac{x-1}{x} + \ln x\right]$

$= x^{x-2}(x - 1 + x\ln x)$

65. $y = (x-2)^{x+1}$

$\ln y = (x+1)\ln(x-2)$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = (x+1)\left(\frac{1}{x-2}\right) + \ln(x-2)$

$\frac{dy}{dx} = y\left[\frac{x+1}{x-2} + \ln(x-2)\right]$

$= (x-2)^{x+1}\left[\frac{x+1}{x-2} + \ln(x-2)\right]$

66. $y = (1+x)^{1/x}$

$\ln y = \frac{1}{x}\ln(1+x)$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x}\left(\frac{1}{1+x}\right) + \ln(1+x)\left(-\frac{1}{x^2}\right)$

$\frac{dy}{dx} = \frac{y}{x}\left[\frac{1}{x+1} - \frac{\ln(x+1)}{x}\right]$

$= \frac{(1+x)^{1/x}}{x}\left[\frac{1}{x+1} - \frac{\ln(x+1)}{x}\right]$

67. $y = x^{\sin x}, \quad \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\ln y = \sin x \ln x$

$\frac{y'}{y} = \frac{\sin x}{x} + \cos x \ln x$

At $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$: $\frac{y'}{(\pi/2)} = \frac{1}{(\pi/2)} + 0$

$y' = 1$

Tangent line: $y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)$

$y = x$

68. $y = (\sin x)^{2x}, \quad \left(\frac{\pi}{2}, 1\right)$

$\ln y = 2x \ln(\sin x)$

$\frac{y'}{y} = \frac{2x}{\sin x} \cos x + 2 \ln(\sin x)$

At $\left(\frac{\pi}{2}, 1\right)$, $y' = 0$.

Tangent line: $y = 1$

69. $y = (\ln x)^{\cos x}, \quad (e, 1)$

$\ln y = \cos x \cdot \ln(\ln x)$

$\frac{y'}{y} = \cos x \cdot \frac{1}{x \ln x} - \sin x \cdot \ln(\ln x)$

At $(e, 1)$, $y' = \cos(e)\frac{1}{e} - 0$.

Tangent line: $y - 1 = \frac{\cos e}{e}(x - e)$

$y = \frac{\cos e}{e}x + 1 - \cos e$

70. $y = x^{1/x}, \quad (1, 1)$

$\ln y = \frac{1}{x} \ln x$

$\frac{y'}{y} = \frac{1}{x^2} - \frac{\ln x}{x^2}$

At $(1, 1)$, $y' = 1 - 0 = 1$.

Tangent line: $y - 1 = 1(x - 1)$

$y = x$

71. $\int 3^x \, dx = \frac{3^x}{\ln 3} + C$

72. $\int 8^{-x} \, dx = -\frac{8^{-x}}{\ln 8} + C$

73. $\int \left(x^2 + 2^{-x}\right) dx = \frac{x^3}{3} + \frac{-1}{\ln 2} 2^{-x} + C$

$= \frac{1}{3}x^3 - \frac{2^{-x}}{\ln 2} + C$

74. $\int \left(x^4 + 5^x\right) dx = \frac{x^5}{5} + \frac{5^x}{\ln 5} + C$

75. $\int x\left(5^{-x^2}\right) dx = -\frac{1}{2}\int 5^{-x^2}(-2x)\, dx$

$= -\left(\frac{1}{2}\right)\frac{5^{-x^2}}{\ln 5} + C$

$= \frac{-1}{2 \ln 5}\left(5^{-x^2}\right) + C$

76. $\int x(x + 4)\, 6^{(x+4)^2}\, dx = \frac{1}{2}\int 6^{(x+4)^2}\, 2(x + 4)\, dx$

$= \left(\frac{1}{2}\right)\frac{6^{(x+4)^2}}{\ln 6} + C$

$= \frac{6^{(x+4)^2}}{2 \ln 6} + C$

77. $\int \frac{3^{2x}}{1 + 3^{2x}}\, dx, u = 1 + 3^{2x}, du = 2(\ln 3)3^{2x}\, dx$

$\frac{1}{2 \ln 3}\int \frac{(2 \ln 3)3^{2x}}{1 + 3^{2x}}\, dx = \frac{1}{2 \ln 3} \ln\left(1 + 3^{2x}\right) + C$

78. $\int 2^{\sin x} \cos x\, dx, u = \sin x, du = \cos x\, dx$

$\frac{1}{\ln 2} 2^{\sin x} + C$

79. $\int_{-1}^{2} 2^x\, dx = \left[\frac{2^x}{\ln 2}\right]_{-1}^{2} = \frac{1}{\ln 2}\left[4 - \frac{1}{2}\right] = \frac{7}{2 \ln 2} = \frac{7}{\ln 4}$

80. $\int_{-4}^{4} 3^{x/4}\, dx = 4\int_{-4}^{4} 3^{x/4}\left(\frac{1}{4}\, dx\right)$

$= \left[4 \frac{1}{\ln 3} 3^{x/4}\right]_{-4}^{4}$

$= \frac{4}{\ln 3}\left(3 - 3^{-1}\right)$

$= \frac{32}{3 \ln 3}$

81. $\int_0^1 (5^x - 3^x)\,dx = \left[\frac{5^x}{\ln 5} - \frac{3^x}{\ln 3}\right]_0^1$

$= \left(\frac{5}{\ln 5} - \frac{3}{\ln 3}\right) - \left(\frac{1}{\ln 5} - \frac{1}{\ln 3}\right)$

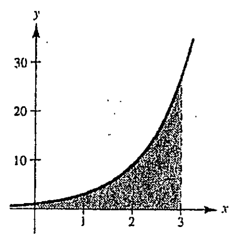
$= \frac{4}{\ln 5} - \frac{2}{\ln 3}$

82. $\int_1^3 (7^x - 4^x)\,dx = \left[\frac{7^x}{\ln 7} - \frac{4^x}{\ln 4}\right]_1^3$

$= \left(\frac{7^3}{\ln 7} - \frac{4^3}{\ln 4}\right) - \left(\frac{7}{\ln 7} - \frac{4}{\ln 4}\right)$

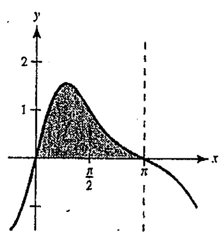
$= \frac{336}{\ln 7} - \frac{60}{\ln 4}$

83. Area $= \int_0^3 3^x\,dx = \left[\frac{1}{\ln 3}3^x\right]_0^3 = \frac{1}{\ln 3}(27 - 1) = \frac{26}{\ln 3} \approx 23.6662$



84. Area $= \int_0^\pi 3^{\cos x}\sin x\,dx$

$= \left[\frac{-3^{\cos x}}{\ln 3}\right]_0^\pi = \frac{-1}{\ln 3}\left[3^{-1} - 3\right] = \frac{8}{3\ln 3} \approx 2.4273$



85. $f(x) = \log_{10} x$

(a) Domain: $x > 0$

(b) $y = \log_{10} x$

$10^y = x$

$f^{-1}(x) = 10^x$

(c) $\log_{10} 1000 = \log_{10} 10^3 = 3$

$\log_{10} 10{,}000 = \log_{10} 10^4 = 4$

If $1000 \le x \le 10{,}000$, then $3 \le f(x) \le 4$.

(d) If $f(x) < 0$, then $0 < x < 1$.

(e) $f(x) + 1 = \log_{10} x + \log_{10} 10 = \log_{10}(10x)$

x must have been increased by a factor of 10.

(f) $\log_{10}\left(\frac{x_1}{x_2}\right) = \log_{10} x_1 - \log_{10} x_2$

$= 3n - n = 2n$

So, $x_1/x_2 = 10^{2n} = 100^n$.

86. $f(x) = \log_2 x \Rightarrow f'(x) = \frac{1}{x \ln 2}$

$g(x) = x^x \Rightarrow g'(x) = x^x(1 + \ln x)$

Note: Let $y = g(x)$. Then:

$\ln y = \ln x^x = x \ln x$

$\frac{1}{y}y' = x \cdot \frac{1}{x} + \ln x$

$y' = y(1 + \ln x)$

$y' = x^x(1 + \ln x) = g'(x)$

$h(x) = x^2 \Rightarrow h'(x) = 2x$

$k(x) = 2^x \Rightarrow k'(x) = (\ln 2)2^x$

From greatest to least rate of growth:

$g(x), k(x), h(x), f(x)$

87. $C(t) = P(1.05)^t$

(a) $C(10) = 24.95(1.05)^{10}$
$\approx \$40.64$

(b) $\dfrac{dC}{dt} = P(\ln 1.05)(1.05)^t$

When $t = 1$, $\dfrac{dC}{dt} \approx 0.051P$.

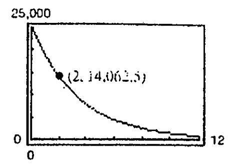
When $t = 8$, $\dfrac{dC}{dt} \approx 0.072P$.

(c) $\dfrac{dC}{dt} = (\ln 1.05)\left[P(1.05)^t\right] = (\ln 1.05)C(t)$

The constant of proportionality is ln 1.05.

88. $V(t) = 25{,}000\left(\dfrac{3}{4}\right)^t$

(a)



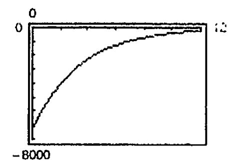
$V(2) = 25{,}000\left(\dfrac{3}{4}\right)^2 = \$14{,}062.50$

(b) $\dfrac{dV}{dt} = 25{,}000\left(\ln\dfrac{3}{4}\right)\left(\dfrac{3}{4}\right)^t$

When $t = 1$, $\dfrac{dV}{dt} \approx -5394.04$.

When $t = 4$, $\dfrac{dV}{dt} \approx -2275.61$.

(c)



Horizontal asymptote: $V' = 0$

As the car ages, it is worth less each year and depreciates less each year, but the value of the car will never reach \$0.

89. $P = \$1000, r = 3\tfrac{1}{2}\% = 0.035, t = 10$

$A = 1000\left(1 + \dfrac{0.035}{n}\right)^{10n}$

$A = 1000e^{(0.035)(10)}$

n	1	2	4	12	365	Continuous
A	\$1410.60	\$1414.78	\$1416.91	\$1418.34	\$1419.04	\$1419.07

90. $P = \$2500, r = 6\% = 0.06, t = 20$

$A = 2500\left(1 + \dfrac{0.06}{n}\right)^{20n}$

$A = 2500e^{(0.06)(20)}$

n	1	2	4	12	365	Continuous
A	\$8017.84	\$8155.09	\$8226.66	\$8275.51	\$8299.47	\$8300.29

118. $y = f(x) = \dfrac{a^x - 1}{a^x + 1}$

$$y' = \frac{(a^x + 1)(a^x \ln a) - (a^x - 1)a^x \ln a}{(a^x + 1)^2}$$

$$= \frac{2a^x \ln a}{(a^x + 1)^2}$$

For $0 < a < 1, y' < 0 \Rightarrow$ one-to-one and has an inverse

For $a > 1, y' > 0 \Rightarrow$ one-to-one and has an inverse

$$y(a^x + 1) = a^x - 1$$

$$a^x(y - 1) = -1 - y$$

$$a^x = \frac{y + 1}{1 - y}$$

$$x \ln a = \ln\left(\frac{y + 1}{1 - y}\right)$$

$$x = \frac{1}{\ln a} \ln\left(\frac{y + 1}{1 - y}\right)$$

$$f^{-1}(x) = \frac{1}{\ln a} \ln\left(\frac{x + 1}{1 - x}\right)$$

119. $$\frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), \quad y(0) = 1$$

$$\frac{dy}{y[(5/4) - y]} = \frac{8}{25}\,dt$$

$$\frac{4}{5}\int\left(\frac{1}{y} + \frac{1}{(5/4) - y}\right)dy = \int\frac{8}{25}\,dt$$

$$\ln y - \ln\left(\frac{5}{4} - y\right) = \frac{2}{5}t + C$$

$$\ln\left(\frac{y}{(5/4) - y}\right) = \frac{2}{5}t + C$$

$$\frac{y}{(5/4) - y} = e^{(2/5)t + C} = C_1 e^{(2/5)t}$$

When $t = 0, y = 1 \Rightarrow C_1 = 4 \Rightarrow 4e^{(2/5)t} = \dfrac{y}{(5/4) - y}$

$$4e^{(2/5)t}\left(\frac{5}{4} - y\right) = y$$

$$5e^{(2/5)t} = 4e^{(2/5)t}y + y = \left(4e^{(2/5)t} + 1\right)y$$

$$y = \frac{5e^{(2/5)t}}{4e^{(2/5)t} + 1} = \frac{5}{4 + e^{-0.4t}} = \frac{1.25}{1 + 0.25e^{-0.4t}}$$

120. $f(x) = a^x$

(a) $f(u + v) = a^{u+v} = a^u a^v = f(u)f(v)$

(b) $f(2x) = a^{2x} = (a^x)^2 = [f(x)]^2$

121. (a)

$$y^x = x^y$$
$$x \ln y = y \ln x$$
$$x\frac{y'}{y} + \ln y = \frac{y}{x} + y' \ln x$$
$$y'\left[\frac{x}{y} - \ln x\right] = \frac{y}{x} - \ln y$$
$$y' = \frac{(y/x) - \ln y}{(x/y) - \ln x}$$
$$y' = \frac{y^2 - xy \ln y}{x^2 - xy \ln x}$$

(b) (i) At (c, c): $y' = \dfrac{c^2 - c^2 \ln c}{c^2 - c^2 \ln c} = 1, (c \neq 0, e)$

(ii) At $(2, 4)$: $y' = \dfrac{16 - 8 \ln 4}{4 - 8 \ln 2} = \dfrac{4 - 4 \ln 2}{1 - 2 \ln 2} \approx -3.1774$

(iii) At $(4, 2)$: $y' = \dfrac{4 - 8 \ln 2}{16 - 8 \ln 4} = \dfrac{1 - 2 \ln 2}{4 - 4 \ln 2} \approx -0.3147$

(c) y' is undefined for

$x^2 = xy \ln x$

$x = y \ln x = \ln x^y$

$e^x = x^y$.

At (e, e), y' is undefined.

122. Let $f(x) = \dfrac{\ln x}{x}, \quad x > 0.$

$f'(x) = \dfrac{1 - \ln x}{x^2} < 0$ for $x > e \Rightarrow f$ is decreasing for $x \geq e$. So, for $e \leq x < y$:

$f(x) > f(y)$

$\dfrac{\ln x}{x} > \dfrac{\ln y}{y}$

$(xy)\dfrac{\ln x}{x} > (xy)\dfrac{\ln y}{y}$

$\ln x^y > \ln y^x$

$x^y > y^x$

For $n \geq 8, e < \sqrt{n} < \sqrt{n+1}, \left(\sqrt{8} \approx 2.828\right)$ and so letting $x = \sqrt{n}, y = \sqrt{n+1}$, you have

$\left(\sqrt{n}\right)^{\sqrt{n+1}} > \left(\sqrt{n+1}\right)^{\sqrt{n}}$.

Note: $\sqrt{8}^{\sqrt{9}} \approx 22.6$ and $\sqrt{9}^{\sqrt{8}} \approx 22.4$.

Note: This same argument shows $e^\pi > \pi^e$.