Key

BC Calculus – 6.10 Notes - Linear Partial Fractions

Let $F(x) = \frac{P(x)}{Q(x)}$ be a rational function where $P(x)$ and $Q(x)$ are polynomials. If the degree of the numerator is smaller than the degree of the denominator (proper rational function), then you might be able to use partial fraction decomposition to change the integrand to something easier to integrate. Let's "decompose" this fraction to make it easier to integrate.

Partial Fraction Decomposition * Degree in denominator is higher

Every proper rational function can be written as a sum

$$\frac{P(x)}{Q(x)} = F_1(x) + F_2(x) + F_3(x) + \cdots + F_n(x).$$

Where $F_1(x), F_2(x), F_3(x), \ldots, F_n(x)$ are also rational functions, in which the denominators are factors of $Q(x)$. We are only concerned with nonrepeating linear factors.

With $Q(x)$ having n nonrepeating linear factors, we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2)(a_3x + b_3)\ldots(a_nx + b_n)$$

* Make each factor a separate denominator

Therefore, using the process of partial fractions and with $Q(x)$ having n nonrepeating linear factors, we can write $\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b} + \frac{A_2}{a_2x+b} + \frac{A_3}{a_3x+b} + \cdots \frac{A_n}{a_nx+b_n}$

1. Evaluate $\int \frac{1}{(x-1)(x+2)}dx$ $\rightarrow \frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

$x=1$ $\quad$ $x=-2$

* cover-up Method

$A = 1/3 \qquad B = -1/3$

$$\int \frac{1/3}{x-1} + \frac{-1/3}{x+2}\,dx$$

$$\frac{1}{3}\int \frac{1}{x-1}dx - \frac{1}{3}\int \frac{1}{x+2}dx$$

$$\boxed{\frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C \quad \text{or} \quad \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C}$$

2. Evaluate $\int \frac{2x^2-7}{x^3-3x^2-4x}dx$ $\rightarrow x(x^2-3x-4) \rightarrow x(x-4)(x+1)$

$$\frac{2x^2-7}{x(x-4)(x+1)} = \frac{A}{x} + \frac{B}{x-4} + \frac{C}{x+1}$$

$x=0 \qquad x=4 \qquad x=-1$

$A = \frac{-7}{-4} = 7/4$

$C = \frac{-5}{5} = -1$

$B = \frac{25}{20} = 5/4$

$$\int \frac{7/4}{x} + \frac{5/4}{x-4} + \frac{-1}{x+1}\,dx$$

$$7/4\int \frac{1}{x} + \frac{5}{4}\int \frac{1}{x-4}dx - 1\int \frac{1}{x+1}dx$$

$$\boxed{\frac{7}{4}\ln|x| + \frac{5}{4}\ln|x-4| - 1\ln|x+1| + C}$$

Practice Problems

Evaluate using partial fractions.

1. $\int \frac{x-12}{x^2-4x}\,dx$

$$\frac{x-12}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$$

$x=0 \qquad x=4$

$A = \frac{-12}{-4} = 3$

$B = \frac{-8}{4} = -2$

$$\int \frac{3}{x} - \frac{2}{x-4}\,dx$$

$$\boxed{3\ln|x| - 2\ln|x-4| + C}$$

OR

$$\boxed{\ln\left|\frac{x^3}{(x-4)^2}\right| + C}$$

2. $\int \frac{2x}{x^2-4}\,dx$ $\rightarrow (x+2)(x-2)$

$$\frac{2x}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$x=-2 \qquad x=2$

$A = \frac{-4}{-4} = 1$

$B = \frac{4}{4} = 1$

$$\int \frac{1}{x+2} + \frac{1}{x-2}\,dx$$

$\ln|x+2| + \ln|x-2| + C$

OR

$\ln|x^2-4| + C$

3. $\int \frac{1}{(x+2)(x-3)(x+1)}\,dx$

$$\frac{1}{(x+2)(x-3)(x+1)} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{x+1}$$

$x=-2 \qquad x=3 \qquad x=-1$

$A = 1/5 \qquad B = \frac{1}{20} \qquad C = -1/4$

$$\int \frac{1/5}{x+2} + \frac{1/20}{x-3} - \frac{1/4}{x+1}\,dx$$

$$\frac{1}{5}\int \frac{1}{x+2}\,dx + \frac{1}{20}\int \frac{1}{x-3}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx$$

$$\boxed{\frac{1}{5}\ln|x+2| + \frac{1}{20}\ln|x-3| - \frac{1}{4}\ln|x+1| + C}$$

4. $\int \frac{x+2}{x^2+5x}\,dx$ $\rightarrow x(x+5)$

$$\frac{x+2}{x(x+5)} = \frac{A}{x} + \frac{B}{x+5}$$

$x=0 \qquad x=-5$

$A = 2/5$

$B = \frac{-3}{-5} = 3/5$

$$\int \frac{\frac{2}{5}}{x} + \frac{3/5}{x+5}\,dx$$

$$\frac{2}{5}\int \frac{1}{x}\,dx + \frac{3}{5}\int \frac{1}{x+5}\,dx$$

$$\boxed{\frac{2}{5}\ln|x| + \frac{3}{5}\ln|x+5| + C}$$

* long division first

5. $\int \frac{2}{x(x-2)}\,dx$

$$\frac{2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$x=0 \quad x=2$

$A = \frac{2}{-2} = -1 \qquad B = \frac{2}{2} = 1$

$$\int \frac{-1}{x} + \frac{1}{x-2}\,dx$$

$$\boxed{-\ln|x| + \ln|x-2| + C}$$

or

$$\boxed{\ln\left|\frac{x-2}{x}\right| + c}$$

6. $\int \frac{x^3-11x-15}{x^2-2x-8}\,dx$

$$x^2-2x-8 \overline{)\, x^3+0x^2-11x-15} \quad x+2+\frac{x+1}{x^2-2x-8}$$

$-(x^3-2x^2-8x)$

$2x^2-3x-15$

$-(2x^2-4x-16)$

$1x+1$

$$\int x+2+\frac{x+1}{x^2-2x-8}\,dx$$

$$\frac{x+1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

$x=4 \quad x=-2$

$A = 5/6 \qquad B = -1/-6 = 1/6$

$$\int \frac{5/6}{x-4} + \frac{1/6}{x+2}\,dx$$

$$\boxed{\frac{x^2}{2} + 2x + \frac{5}{6}\ln|x-4| + \frac{1}{6}\ln|x+2| + C}$$

7. For $0 < P < 50$, what is the antiderivative of $\frac{1}{P(50-P)}$?

$$\frac{1}{P(50-P)} = \frac{A}{P} + \frac{B}{50-P}$$

$P=0 \quad P=50$

$A = \frac{1}{50} \qquad B = \frac{1}{50}$

$$\int \frac{1/50}{P} + \frac{1/50}{50-P}$$

$u = 50-P$

$\frac{du}{dP} = -1$

$dP = -1\,du$

$$\frac{1}{50}\int \frac{1}{P}\,dP + \frac{1}{50}\int \frac{1}{50-P}\,dP$$

$$\boxed{\frac{1}{50}\ln|P| - \frac{1}{50}\ln|50-P| + C \quad \text{or} \quad \ln\left|\left(\frac{P}{50-P}\right)^{1/50}\right| + C}$$

8. $\int_0^1 \frac{1}{(x+5)(x+1)}\,dx$

$$\frac{1}{(x+5)(x+1)} = \frac{A}{x+5} + \frac{B}{x+1}$$

$x=-5 \quad x=-1$

$A = \frac{-1}{4} \qquad B = 1/4$

$$\int \frac{-1/4}{x+5} + \frac{1/4}{x+1}\,dx =$$

$$-\frac{1}{4}\ln|x+5| + \frac{1}{4}\ln|x+1| = \frac{1}{4}\ln\left|\frac{x+1}{x+5}\right|\Bigg]_0^1$$

$$\frac{1}{4}\ln\left|\frac{1}{3}\right| - \frac{1}{4}\ln\left|\frac{1}{5}\right|$$

$$= \frac{1}{4}\ln\left|\frac{1/3}{1/5}\right| = \frac{1}{4}\ln\left|\frac{1}{3}\cdot\frac{5}{1}\right| = \boxed{\frac{1}{4}\ln\left|5/3\right|}$$

9. $\int_0^2 \frac{3}{(4x+1)(x+1)}\,dx$

$$\frac{3}{(4x+1)(x+1)} = \frac{A}{4x+1} + \frac{B}{x+1}$$

$x = -1/4 \quad x = -1$

$$A = \frac{3}{3/4} = 3 \cdot 4/3 = 4$$

$$B = \frac{3}{-3} = -1$$

$$\int \frac{4}{4x+1} - \frac{1}{x+1}\,dx$$

$u = 4x+1$ | $dx = \frac{du}{4}$

$\frac{du}{dx} = 4$

$$4\int \frac{1}{u} \cdot \frac{du}{4} - \int \frac{1}{x+1}\,dx$$

$$\ln|4x+1| - \ln|x+1| \Big]_0^2$$

$$\ln(9) - \ln(3) - (\ln 1 - \ln 1)$$

$$= \ln 9 - \ln 3 = \ln\left(\frac{9}{3}\right)$$

$$= \boxed{\ln 3}$$

10. $\int_2^3 \frac{3}{(x-1)(x+2)}\,dx$

$$\frac{3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$x = 1 \quad x = -2$

$A = \frac{3}{3} = 1 \qquad B = \frac{3}{-3} = -1$

$$\int \frac{1}{x-1} - \frac{1}{x+2}\,dx$$

$$\ln|x-1| - \ln|x+2| \Big]_2^3$$

$$\ln 2 - \ln 5 - (\ln 1 - \ln 4)$$

$$\ln 2 - \ln 5 + \ln 4$$

$$= \ln\left(\frac{2 \cdot 4}{5}\right) = \boxed{\ln\left(\frac{8}{5}\right)}$$