Key

6.12 AP Practice Problems (p. 526) - Improper Integrals

1. $\int_0^2 \frac{x+2}{x^2+4x-12}\,dx =$

(A) $-\frac{\ln 12}{2}$ (B) $\frac{1-\ln 12}{2}$

(C) $\frac{\ln 12 - \ln 2}{2}$ (D) diverges

$$\int \frac{\cancel{x+2}}{u}\cdot\frac{du}{2(\cancel{x+2})}$$

$$\frac{1}{2}\int\frac{1}{u}du = \frac{1}{2}\ln|u|$$

$$\int_0^2 \frac{x+2}{(x+6)(x-2)}$$

VA at $x=2$

$$\lim_{b\to 2^-}\int_0^b \frac{x+2}{x^2+4x-12}\,dx$$

$u = x^2+4x-12$

$\frac{du}{dx} = 2x+4 = 2(x+2)$

$dx = \frac{du}{2(x+2)}$

$$\lim_{b\to 2^-} \frac{1}{2}\ln|x^2+4x-12|\Big]_0^b$$

$$\frac{1}{2}\ln|b^2+4b-12| - \frac{1}{2}\ln 12$$

$= -\infty \rightarrow$ so $\boxed{\text{diverges}}$

(ex: $\ln|0.0001| = \boxed{-\infty}$)

2. Determine whether $\int_1^\infty \frac{2}{x^3}\,dx$ converges or diverges. If it converges, find its value.

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) diverges

$$\int 2x^{-3}dx \rightarrow \frac{2x^{-2}}{-2} \rightarrow -\frac{1}{x^2}\Big]_1^b$$

$$\lim_{b\to\infty} \frac{-1}{b^2} - \left(\frac{-1}{1}\right) = 0+1 = \boxed{1}$$

3. Determine whether $\int_3^\infty \frac{8x}{\sqrt[3]{8-x^2}}\,dx$ converges or diverges. If it converges, find its value.

(A) 0 (B) 18 (C) $6(9^{2/3})$ $\boxed{\text{(D)}}$ diverges

$\int \frac{8x}{(8-x^2)^{1/3}}\,dx$

$u=8-x^2$

$\frac{du}{dx}=-2x$

$dx=\frac{du}{-2x}$

$\int \frac{8x}{u^{1/3}}\cdot\frac{du}{-2x}$

$-4\int u^{-1/3}du$

$-4\left(\frac{u^{2/3}}{2/3}\right)$

$-4\cdot\frac{3}{2}u^{2/3}$

$-6(8-x^2)^{2/3}\Big]_3^b$

$\lim_{b\to\infty} -6(8-b^2)^{2/3}-\left(-6(8-9)^{2/3}\right) = \infty$

$\boxed{\text{diverges}}$

4. Determine whether $\int_{-\infty}^0 xe^{x^2}\,dx$ converges or diverges. If it converges, find its value.

(A) $\frac{1}{2}$ (B) 1 (C) $-e$ $\boxed{\text{(D)}}$ diverges

$\int xe^{x^2}dx$

$u=x^2$

$\frac{du}{dx}=2x$

$dx=\frac{du}{2x}$

$\int xe^u\cdot\frac{du}{2x}$

$\frac{1}{2}\int e^u du$

$\frac{1}{2}e^u$

$\lim_{b\to-\infty}\frac{1}{2}e^{x^2}\Big]_b^0$

$\lim_{b\to-\infty}\frac{1}{2}e^0-\frac{1}{2}e^{b^2} = \frac{1}{2}-\infty = -\infty$

$\boxed{\text{Diverges}}$

5. Determine whether $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$ converges or diverges. If it converges, find its value.

$\boxed{\text{(A)}}$ $\frac{1}{\ln 2}$ (B) $\frac{1}{2}$ (C) $-\frac{1}{\ln 2}$ (D) diverges

$$\int \frac{1}{x(\ln x)^2}\,dx$$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

$$\int \frac{1}{\cancel{x}\cdot u^2}\cdot \cancel{x}\,du$$

$$\int u^{-2}\,du \rightarrow \frac{u^{-1}}{-1} \rightarrow -\frac{1}{u} \rightarrow \frac{-1}{\ln x}$$

$$\lim_{b\to\infty} \left.\frac{-1}{\ln x}\right]_2^b \rightarrow \lim_{b\to\infty} \frac{-1}{\ln b} - \left(\frac{-1}{\ln 2}\right) = 0 + \frac{1}{\ln 2} = \boxed{\frac{1}{\ln 2}}$$

6. Determine whether $\int_0^1 \frac{1}{\sqrt[4]{x}}\,dx$ converges or diverges. If it converges, find its value.

(A) $\frac{3}{4}$ (B) 4 $\boxed{\text{(C)}}$ $\frac{4}{3}$ (D) diverges

VA at $x = 0$

$$\lim_{b\to 0^+} \int_b^1 x^{-1/4}\,dx$$

$$\frac{x^{3/4}}{3/4} \rightarrow \left.\frac{4}{3}x^{3/4}\right]_b^1$$

$$\lim_{b\to 0^+} \frac{4}{3}(1)^{3/4} - \frac{4}{3}(b)^{3/4} = \frac{4}{3} - 0 = \boxed{\frac{4}{3}}$$

7. When the region bounded by the graph of $y = \frac{1}{x^2}$ and the x-axis to the right of the line $x = 1$ is revolved about the x-axis, the volume V, if it is defined, of the solid of revolution that is generated is given by the improper integral

$$\int_1^{\infty} \pi \left(\frac{1}{x^2}\right)^2 dx.$$

Determine whether the improper integral converges or diverges. If it converges, find the volume of the solid of revolution.

$$\pi \int \frac{1}{x^4} dx \rightarrow \int x^{-4} dx \rightarrow \frac{x^{-3}}{-3} \rightarrow -\frac{1}{x^3}\Big]_1^b$$

$$\lim_{b \to \infty} -\frac{1}{b^3} - \left(-\frac{1}{(1)^3}\right) = 0 + \frac{1}{3} \rightarrow \pi\left(\frac{1}{3}\right) = \boxed{\frac{\pi}{3}}$$

$\boxed{\text{Converges to } \pi/3}$