Key

BC Calculus – 6.12 Notes – Improper Integrals

Improper integrals are integrals with infinite limits of integration or have an infinite discontinuity on the interval.

$\boxed{\text{Examples}}$: $\int_1^{\infty} \frac{1}{x}\,dx$ or $\int_{-1}^{1} \frac{1}{x}\,dx$ ← vertical asymptote at $x=0$ inside the interval

If $f(x)$ is continuous on $[a,\infty)$, then $\int_a^{\infty} f(x)\,dx =$ $\lim_{b\to\infty} \int_a^b f(x)\,dx$

If $f(x)$ is continuous on $(-\infty, b]$, then $\int_{-\infty}^{b} f(x)\,dx =$ $\lim_{a\to-\infty} \int_a^b f(x)\,dx$

Provided the limits exist!

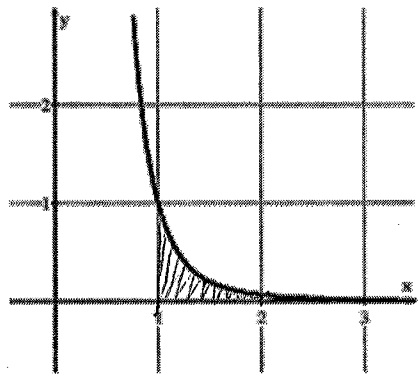
*Instead of upper bound of infinity, we will approach infinity

1. $\int_1^{\infty} \frac{1}{x^4}\,dx$

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^4}\,dx \rightarrow \lim_{b\to\infty} \int_1^b x^{-4}\,dx$$

$$\frac{x^{-3}}{-3} \rightarrow \left.\frac{-1}{3x^3}\right]_1^b \rightarrow \lim_{b\to\infty} \frac{-1}{3b^3} - \left(\frac{-1}{3}\right)$$

$$0 + \frac{1}{3} = \boxed{\frac{1}{3}}$$ ← Area "converges" to $\frac{1}{3}$

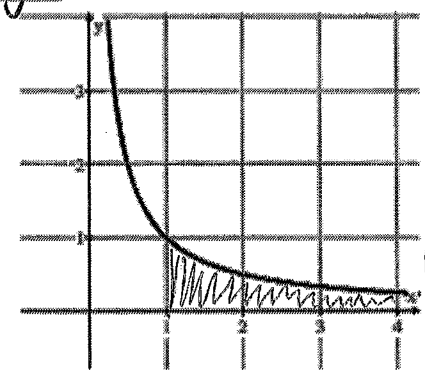


If the limit exists, the improper integral is said to converge**. If the limit does not exist, the integral is said to** diverge**.**

2. $\int_1^{\infty} \frac{1}{x}\,dx$

$$\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx \rightarrow \left.\ln|x|\right]_1^b$$

$$\lim_{b\to\infty} \ln|b| - \ln|1| = \infty - 0 \quad \boxed{\text{diverges}}$$



*Is the graph closing in on the x-axis fast enough?

Improper *p-integral*: $\int_1^{\infty} \frac{1}{x^p}\,dx$ **converges if** $\boxed{p>1}$ **and diverges if** $\boxed{p\le 1}$

Remember the definite integral $\int_a^b f(x)\,dx$, requires the interval to be finite and the FTC requires that $f(x)$ be continuous on $[a, b]$. If the integral does not meet these requirements, we may need to manipulate the problem.

Another form of the Improper Integral is $\int_{-\infty}^{\infty} f(x)\,dx$, with $f(x)$ continuous on $(-\infty, \infty)$. Let $x = c$ be any real number in the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

*we want to anchor one of the integral bounds with a Real number.

(It's easiest to use 0 here for c). If either of these integrals diverge, then the whole diverges.

3. $\int_{-\infty}^{\infty} e^x\,dx$

$$\int_{-\infty}^{0} e^x\,dx + \int_0^{\infty} e^x\,dx$$

$$\lim_{a\to-\infty}\int_a^0 e^x\,dx + \lim_{b\to\infty}\int_0^b e^x\,dx$$

$$\lim_{a\to-\infty} e^x\Big]_a^0 + \lim_{b\to\infty} e^x\Big]_0^b$$

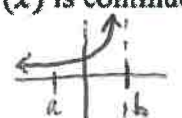
$$\lim_{a\to-\infty} e^0 - e^a + \lim_{b\to\infty} e^b - e^0$$

$$\left(1 - \frac{1}{e^{\infty}}\right) + \underbrace{\left(e^{\infty} - 1\right)}$$

this portion diverges so the whole integral diverges.

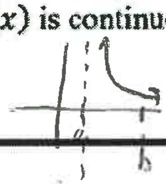
$\boxed{\text{Diverges}}$

If $f(x)$ is continuous on $[a, b)$ and has an infinite discontinuity at b, then (vertical asymptote at $x = b$)



$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

If $f(x)$ is continuous on $(a, b]$ and has an infinite discontinuity at a, then (vertical asymptote at $x = a$)



$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

4. $\int_0^2 \frac{x+2}{\sqrt{x^2+4x}}\,dx$

$$\frac{x+2}{\sqrt{x(x+4)}}$$

*discontinuity at $x=0$ and $x=4$

$$\lim_{t\to0^-}\int_t^2 \frac{x+2}{(x^2+4x)^{1/2}}\,dx$$

$u = x^2+4x$

$\frac{du}{dx} = 2x+4$

$dx = \frac{du}{2x+4}$

$$\lim_{t\to0^-}\int_t^2 \frac{\cancel{x+2}}{u^{1/2}}\cdot\frac{du}{2(\cancel{x+2})}$$

$$\frac{1}{2}\int u^{-1/2}\,du$$

$$\frac{1}{2}\cdot\frac{u^{1/2}}{1/2} \to u^{1/2} \to (x^2+4x)^{1/2}\Big]_t^2$$

$$\lim_{t\to0^-} (12)^{1/2} - (t)^{1/2} = \boxed{\sqrt{12}}$$

If $f(x)$ is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

* These interior discontinuities are easy to miss!

5. $\int_{-1}^{1} \frac{1}{x}\,dx$ → Interior discontinuity at $x=0$ (VA at $x=0$)

$$\int_{-1}^{0}\frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx$$

$$\lim_{t\to0^-}\int_{-1}^{t}\frac{1}{x}\,dx + \lim_{t\to0^+}\int_t^1 \frac{1}{x}\,dx$$

$$\lim_{t\to0^-} \ln|x|\Big]_{-1}^{t} + \lim_{t\to0^+} \ln|x|\Big]_t^1$$

$$\ln|t| - \ln|1| + \ln|1| - \ln|t|$$

diverges to $-\infty$

$\boxed{\text{Diverges}}$

Evaluate each integral.

1. $\int_1^\infty \frac{1}{x^2}\,dx$

$$\lim_{b\to\infty}\int_1^b x^{-2}\,dx \rightarrow \left.\frac{x^{-1}}{-1}\right]_1^b$$

$$\lim_{b\to\infty} \frac{-1}{b} - \left(-\frac{1}{1}\right) = 0 + 1 = \boxed{1}$$

2. $\int_0^\infty \frac{2}{x^2+4x+3}\,dx$ * partial fraction

$$\int \frac{2}{(x+3)(x+1)}\,dx \rightarrow \frac{A}{x+3} + \frac{B}{x+1} \quad A=-1 \quad B=\frac{2}{2}=1$$

$x=-3$ $\quad$ $x=-1$

$$\int \frac{-1}{x+3} + \frac{1}{x+1}\,dx$$

$$-\ln|x+3| + \ln|x+1|\Big]_0^b \rightarrow \ln\left|\frac{x+1}{x+3}\right|\Bigg]_0^b$$

$$\lim_{b\to\infty} \ln\left|\frac{b+1}{b+3}\right| - \ln\left|\frac{0+1}{0+3}\right|$$

$$= \ln(1) - \ln\left(\frac{1}{3}\right)$$

$$= \ln 1 - (\ln 1 - \ln 3)$$

$$= 0 - 0 + \ln 3 = \boxed{\ln 3}$$

3. $\int_0^1 \frac{x+1}{\sqrt{x^2+2x}}\,dx$ * discontinuous at $x=0$

$$\int_0^1 \frac{x+1}{\sqrt{x(x+2)}}\,dx$$

$$u = x^2+2x$$

$$\frac{du}{dx} = 2x+2$$

$$dx = \frac{du}{2(x+1)}$$

$$\int \frac{\cancel{x+1}}{u^{1/2}} \cdot \frac{du}{2(\cancel{x+1})}$$

$$\frac{1}{2}\int u^{-1/2}\,du$$

$$\frac{1}{2}\cdot\frac{u^{1/2}}{1/2} \rightarrow u^{1/2}$$

$$(x^2+2x)^{1/2}\Big]_b^1$$

$$\lim_{b\to 0^+} (1+2)^{1/2} - (b^2+2b)^{1/2}$$

$$= 3^{1/2}$$

$$= \boxed{\sqrt{3}}$$

4. $\int_1^\infty xe^{-x}\,dx$ * Integration By Parts

$f = x \quad g' = e^{-x}$

$f' = 1 \quad g = -e^{-x}$

$$-xe^{-x} - \int(-e^{-x})\,dx$$

$$-xe^{-x} + (-e^{-x})\Big]_1^b$$

$$\lim_{b\to\infty} \frac{-b}{e^b} - \frac{1}{e^b} - \left(\frac{-1}{e^1} - \frac{1}{e^1}\right)$$

$$0 - 0 + \frac{2}{e}$$

$$= \boxed{\frac{2}{e}}$$

5. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$\arctan\left(\frac{x}{1}\right)\Big]_a^0 + \arctan\left(\frac{x}{1}\right)\Big]_0^b$$

$$\lim_{a\to-\infty} \arctan(0) - \arctan(a) + \lim_{b\to\infty} \arctan(b) - \arctan(0)$$

$$0 - (-\pi/2) + (\pi/2) - 0$$

$$\frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$

6. $\int_{-1}^{0} \frac{1}{x^5}\,dx$ ← VA at $x = 0$

$$\int_{-1}^{0} x^{-5}\,dx$$

$$\lim_{b\to0^-} \frac{x^{-4}}{-4}\Big]_{-1}^{b}$$

$$\lim_{b\to0^-} \frac{-1}{4b^4} - \left(\frac{-1}{4(-1)^4}\right)$$

$$-\infty + \frac{1}{4} = -\infty$$

$$\boxed{\text{Diverges}}$$

7. $\int_{0}^{\infty} e^{-x}\,dx$

$$\lim_{b\to\infty} -e^{-x}\Big]_0^b$$

$$\lim_{b\to\infty} \frac{-1}{e^b} - \left(-\frac{1}{e^0}\right)$$

$$0 + \frac{1}{1} = \boxed{1}$$

8. Determine all the values of p for which $\int_{0}^{1} \frac{1}{x^p}\,dx$ converges.

$$\lim_{b\to0^+} \int x^{-p}\,dx$$

$$\lim_{b\to0^+} \frac{x^{-p+1}}{1-p}\Big]_b^1 \rightarrow \frac{1}{x^{(p-1)}(1-p)}\Big]_b^1$$

$$\lim_{b\to0^+} \frac{1}{1^{(p-1)}(1-p)} - \frac{1}{b^{(p-1)}(1-p)}$$

$$\lim_{b\to0^+} \frac{1}{1-p} - \left(\frac{1}{b^{p-1}(1-p)}\right)$$

As $b\to0^+$, $p-1$ must be negative for Integral to converge. $p-1<0$

$\boxed{\text{therefore } p<1}$

9. If g is a twice-differentiable function, where $g(2) = 1$ and $\lim_{x\to\infty} g(x) = 8$, then $\int_2^\infty g'(x)\,dx$ is

$$\lim_{b\to\infty} \int_2^b g'(x)\,dx = \lim_{b\to\infty} g(b) - g(2)$$

$$8 - 1 = \boxed{7}$$

A) -7 $\quad$ $\boxed{\text{(B) } 7}$ $\quad$ (C) 9 $\quad$ (D) nonexistent

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10. If R is the unbounded region between the graph of $y = \frac{x}{(1+x^2)^2}$ and the x-axis for $x \geq 0$, then the area of R is

$$\int_0^\infty \frac{x}{(1+x^2)^2}\,dx$$

$u = 1+x^2$ $\quad$ $dx = \frac{du}{2x}$

$\frac{du}{dx} = 2x$

$$\int \frac{\cancel{x}}{u^2} \cdot \frac{du}{2\cancel{x}}$$

$$\frac{1}{2}\int u^{-2}\,du$$

$$\frac{1}{2} \cdot \frac{u^{-1}}{-1} \rightarrow \left.\frac{-1}{2(1+x^2)}\right]_0^b$$

$$\lim_{b\to\infty} \frac{-1}{2(1+b^2)} - \left(\frac{-1}{2(1)}\right)$$

$$0 + \frac{1}{2} = \boxed{\frac{1}{2}}$$

A) -1 $\quad$ (B) 0 $\quad$ $\boxed{\text{(C) } \frac{1}{2}}$ $\quad$ (D) infinite

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11. For what values of p will $\int_1^\infty \frac{1}{x^{7p-3}}\,dx$ converge?

converges if $7p - 3 > 1$

$7p > 4$

$\boxed{p > 4/7}$

A) $p < 0$ $\quad$ (B) $p > 0$ $\quad$ $\boxed{\text{(C) } p > \frac{4}{7}}$ $\quad$ (D) $p < \frac{4}{7}$