98.

$$f(x) + f''(x) = -xg(x)f'(x), \qquad g(x) \ge 0$$

$$2f(x)f'(x) + 2f'(x)f''(x) = -2xg(x)[f'(x)]^2$$

$$\frac{d}{dx}\left[f(x)^2 + f'(x)^2\right] = -2x\,g(x)[f'(x)]^2$$

For $x < 0, -2x\,g(x)[f'(x)]^2 \ge 0$

For $x > 0, -2x\,g(x)[f'(x)]^2 \le 0$

So, $f(x)^2 + f'(x)^2$ is increasing for $x < 0$ and decreasing for $x > 0$.

$f(x)^2 + f'(x)^2$ has a maximum at $x = 0$. So, it is bounded by its value at $x = 0$, $f(0)^2 + f'(0)^2$. So, f (and f') is bounded.

99. Let the vertical line $x = k$ cut the graph of the solution $y = f(x)$ at (k, t). The tangent line at (k, t) is

$$y - t = f'(k)(x - k)$$

Because $y' + p(x)y = q(x)$, you have

$$y - t = [q(k) - p(k)t](x - k)$$

For any value of t, this line passes through the point $\left(k + \frac{1}{p(k)}, \frac{q(k)}{p(k)}\right)$.

To see this, note that

$$\frac{q(k)}{p(k)} - t \stackrel{?}{=} [q(k) - p(k)t]\left(k + \frac{1}{p(k)} - k\right)$$

$$\stackrel{?}{=} q(k)k - p(k)tk + \frac{q(k)}{p(k)} - t - kq(k) + p(k)kt = \frac{q(k)}{p(k)} - t.$$

Section 6.2 Differential Equations: Growth and Decay

1. $\frac{dy}{dx} = x + 3$

$$y = \int(x + 3)\,dx = \frac{x^2}{2} + 3x + C$$

2. $\frac{dy}{dx} = 5 - 8x$

$$y = \int(5 - 8x)\,dx = 5x - 4x^2 + C$$

3.

$$\frac{dy}{dx} = y + 3$$

$$\frac{dy}{y + 3} = dx$$

$$\int\frac{1}{y + 3}\,dy = \int dx$$

$$\ln|y + 3| = x + C_1$$

$$y + 3 = e^{x + C_1} = Ce^x$$

$$y = Ce^x - 3$$

4.

$$\frac{dy}{dx} = 6 - y$$

$$\frac{dy}{6 - y} = dx$$

$$\int\frac{-1}{6 - y}\,dy = \int -dx$$

$$\ln|6 - y|dy = -x + C_1$$

$$6 - y = e^{-x + C_1} = Ce^{-x}$$

$$y = 6 - Ce^{-x}$$

5.

$$y' = \frac{5x}{y}$$

$$yy' = 5x$$

$$\int yy'\,dx = \int 5x\,dx$$

$$\int y\,dy = \int 5x\,dx$$

$$\frac{1}{2}y^2 = \frac{5}{2}x^2 + C_1$$

$$y^2 - 5x^2 = C$$

98. $f(x) + f''(x) = -xg(x)f'(x), \qquad g(x) \ge 0$

$$2f(x)f'(x) + 2f'(x)f''(x) = -2xg(x)[f'(x)]^2$$

$$\frac{d}{dx}\left[f(x)^2 + f'(x)^2\right] = -2x\,g(x)[f'(x)]^2$$

For $x < 0, -2x\,g(x)[f'(x)]^2 \ge 0$

For $x > 0, -2x\,g(x)[f'(x)]^2 \le 0$

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$$\ln|y + 3| = x + C_1$$

$$y + 3 = e^{x + C_1} = Ce^x$$

$$y = Ce^x - 3$$

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$$\int \frac{-1}{6 - y}\,dy = \int -dx$$

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$$6 - y = e^{-x + C_1} = Ce^{-x}$$

$$y = 6 - Ce^{-x}$$

5. $y' = \frac{5x}{y}$

$$yy' = 5x$$

$$\int yy'\,dx = \int 5x\,dx$$

$$\int y\,dy = \int 5x\,dx$$

$$\frac{1}{2}y^2 = \frac{5}{2}x^2 + C_1$$

$$y^2 - 5x^2 = C$$

6.
$$y' = -\frac{\sqrt{x}}{4y}$$
$$4y\,y' = -\sqrt{x}$$
$$\int 4y\,dy = \int -\sqrt{x}\,dx$$
$$2y^2 = -\frac{2}{3}x^{3/2} + C_1$$
$$6y^2 + 2x^{3/2} = C$$

7.
$$y' = \sqrt{xy}$$
$$\frac{y'}{y} = \sqrt{x}$$
$$\int \frac{y'}{y}\,dx = \int \sqrt{x}\,dx$$
$$\int \frac{dy}{y} = \int \sqrt{x}\,dx$$
$$\ln|y| = \frac{2}{3}x^{3/2} + C_1$$
$$y = e^{(2/3)x^{3/2} + C_1}$$
$$= e^{C_1}e^{(2/3)x^{3/2}}$$
$$= Ce^{\left(2x^{3/2}\right)/3}$$

8.
$$y' = x(1 + y)$$
$$\frac{y'}{1 + y} = x$$
$$\int \frac{y'}{1 + y}\,dx = \int x\,dx$$
$$\int \frac{dy}{1 + y} = \int x\,dx$$
$$\ln(1 + y) = \frac{x^2}{2} + C_1$$
$$1 + y = e^{\left(x^2/2\right) + C_1}$$
$$y = e^{C_1}e^{x^2/2} - 1$$
$$= Ce^{x^2/2} - 1$$

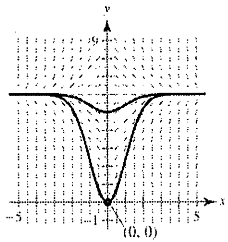
9. $\left(1 + x^2\right)y' - 2xy = 0$
$$y' = \frac{2xy}{1 + x^2}$$
$$\frac{y'}{y} = \frac{2x}{1 + x^2}$$
$$\int \frac{y'}{y}\,dx = \int \frac{2x}{1 + x^2}\,dx$$
$$\int \frac{dy}{y} = \int \frac{2x}{1 + x^2}\,dx$$
$$\ln|y| = \ln\left(1 + x^2\right) + C_1$$
$$\ln|y| = \ln\left(1 + x^2\right) + \ln C$$
$$\ln|y| = \ln\left[C\left(1 + x^2\right)\right]$$
$$y = C\left(1 + x^2\right)$$

10.
$$xy + y' = 100x$$
$$y' = 100x + xy = x(100 - y)$$
$$\frac{y'}{100 - y} = x$$
$$\int \frac{y'}{100 - y}\,dx = \int x\,dx$$
$$\int \frac{1}{100 - y}\,dy = \int x\,dx$$
$$-\ln(100 - y) = \frac{x^2}{2} + C_1$$
$$\ln(100 - y) = -\frac{x^2}{2} - C_1$$
$$100 - y = e^{-\left(x^2/2\right) - C_1}$$
$$-y = e^{-C_1}e^{-x^2/2} - 100$$
$$y = 100 - Ce^{-x^2/2}$$

11.
$$\frac{dQ}{dt} = \frac{k}{t^2}$$
$$\int \frac{dQ}{dt}\,dt = \int \frac{k}{t^2}\,dt$$
$$\int dQ = -\frac{k}{t} + C$$
$$Q = -\frac{k}{t} + C$$

12.
$$\frac{dP}{dt} = k(25 - t)$$
$$\int \frac{dP}{dt}\,dt = \int k(25 - t)\,dt$$
$$\int dP = -\frac{k}{2}(25 - t)^2 + C$$
$$P = -\frac{k}{2}(25 - t)^2 + C$$

13. (a)



(b) $\dfrac{dy}{dx} = x(6 - y), \quad (0, 0)$

$$\frac{dy}{y - 6} = -x\,dx$$

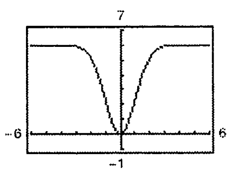
$$\ln|y - 6| = \frac{-x^2}{2} + C$$

$$y - 6 = e^{-x^2/2 + C} = C_1 e^{-x^2/2}$$

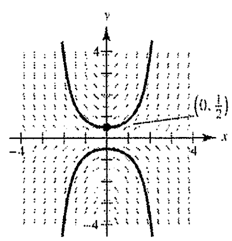
$$y = 6 + C_1 e^{-x^2/2}$$

$$(0, 0):\ 0 = 6 + C_1 \Rightarrow C_1 = -6$$

$$y = 6 - 6e^{-x^2/2}$$



14. (a)



(b) $\dfrac{dy}{dx} = xy, \quad \left(0, \dfrac{1}{2}\right)$

$$\frac{dy}{y} = x\,dx$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$y = e^{x^2/2 + C} = C_1 e^{x^2/2}$$

$$\left(0, \frac{1}{2}\right):\ \frac{1}{2} = C_1 e^0 \Rightarrow C_1 = \frac{1}{2}$$

$$y = \frac{1}{2} e^{x^2/2}$$

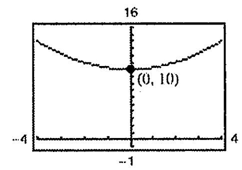
15. $\dfrac{dy}{dt} = \dfrac{1}{2}t, \quad (0, 10)$

$$\int dy = \int \frac{1}{2}t\,dt$$

$$y = \frac{1}{4}t^2 + C$$

$$10 = \frac{1}{4}(0)^2 + C \Rightarrow C = 10$$

$$y = \frac{1}{4}t^2 + 10$$



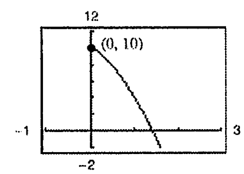
16. $\dfrac{dy}{dt} = -9\sqrt{t}, \quad (0, 10)$

$$\int dy = \int -9\sqrt{t}\,dt$$

$$y = -6t^{3/2} + C$$

$$10 = 0 + C \Rightarrow C = 10$$

$$y = -6t^{3/2} + 10$$



17. $\dfrac{dy}{dt} = -\dfrac{1}{2}y, \quad (0, 10)$

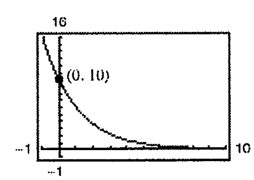
$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dt$$

$$\ln|y| = -\frac{1}{2}t + C_1$$

$$y = e^{-(t/2) + C_1} = e^{C_1}e^{-t/2} = Ce^{-t/2}$$

$$10 = Ce^0 \Rightarrow C = 10$$

$$y = 10e^{-t/2}$$



18. $\frac{dy}{dt} = \frac{3}{4}y, \quad (0, 10)$

$$\int \frac{dy}{y} = \int \frac{3}{4}\, dt$$

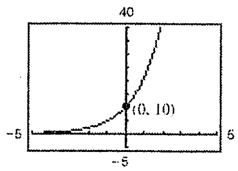
$$\ln y = \frac{3}{4}t + C_1$$

$$y = e^{(3/4)t + C_1}$$

$$= e^{C_1}e^{(3/4)t} = Ce^{3t/4}$$

$$10 = Ce^0 \Rightarrow C = 10$$

$$y = 10e^{3t/4}$$



19. $\frac{dN}{dt} = kN$

$N = Ce^{kt}$ (Theorem 6.1)

$(0, 250)$: $C = 250$

$(1, 400)$: $400 = 250e^k \Rightarrow k = \ln\frac{400}{250} = \ln\frac{8}{5}$

$$N = 250e^{\ln(8/5)t} \approx 250e^{0.4700t}$$

When $t = 4$, $N = 250e^{4\ln(8/5)} = 250e^{\ln(8/5)^4}$

$$= 250\left(\frac{8}{5}\right)^4 = \frac{8192}{5}.$$

20. $\frac{dP}{dt} = kP$

$P = Ce^{kt}$ (Theorem 6.1)

$(0, 5000)$: $C = 5000$

$(1, 4750)$: $4750 = 5000e^k \Rightarrow k = \ln\left(\frac{19}{20}\right)$

$$P = 5000e^{\ln(19/20)t} \approx 5000e^{-0.0513t}$$

When $t = 5$, $P = 5000e^{\ln(19/20)(5)}$

$$= 5000\left(\frac{19}{20}\right)^5 \approx 3868.905.$$

21. $y = Ce^{kt}, \quad \left(0, \frac{1}{2}\right), (5, 5)$

$$C = \frac{1}{2}$$

$$y = \frac{1}{2}e^{kt}$$

$$5 = \frac{1}{2}e^{5k}$$

$$k = \frac{\ln 10}{5}$$

$$y = \frac{1}{2}e^{[(\ln 10)/5]t} = \frac{1}{2}\left(10^{t/5}\right) \text{ or } y \approx \frac{1}{2}e^{0.4605t}$$

22. $y = Ce^{kt}, \quad (0, 4), \left(5, \frac{1}{2}\right)$

$$C = 4$$

$$y = 4e^{kt}$$

$$\frac{1}{2} = 4e^{5k}$$

$$k = \frac{\ln(1/8)}{5} \approx -0.4159$$

$$y = 4e^{-0.4159t}$$

23. $y = Ce^{kt}, \quad (1, 5), (5, 2)$

$$5 = Ce^k \Rightarrow 10 = 2Ce^k$$

$$2 = Ce^{5k} \Rightarrow 10 = 5Ce^k$$

$$2Ce^k = 5Ce^{5k}$$

$$2e^k = 5e^{5k}$$

$$\frac{2}{5} = e^{4k}$$

$$k = \frac{1}{4}\ln\left(\frac{2}{5}\right) = \ln\left(\frac{2}{5}\right)^{1/4}$$

$$C = 5e^{-k} = 5e^{-1/4\ln(2/5)} = 5\left(\frac{2}{5}\right)^{-1/4} = 5\left(\frac{5}{2}\right)^{1/4}$$

$$y = 5\left(\frac{5}{2}\right)^{1/4}e^{[1/4\ln(2/5)]t} \approx 6.2872\, e^{-0.2291t}$$

24. $y = Ce^{kt}, \quad \left(3, \frac{1}{2}\right), (4, 5)$

$\frac{1}{2} = Ce^{3k} \Rightarrow 1 = 2Ce^{3k}$

$5 = Ce^{4k} \Rightarrow 1 = \frac{1}{5}Ce^{4k}$

$2Ce^{3k} = \frac{1}{5}Ce^{4k}$

$10e^{3k} = e^{4k}$

$10 = e^{k}$

$k = \ln 10 \approx 2.3026$

$y = Ce^{2.3026t}$

$5 = Ce^{2.3026(4)}$

$C \approx 0.0005$

$y = 0.0005e^{2.3026t}$

25. In the model $y = Ce^{kt}$, C represents the initial value of y (when $t = 0$). k is the proportionality constant.

26. $y' = \frac{dy}{dt} = ky$

27. $\frac{dy}{dx} = \frac{1}{2}xy$

$\frac{dy}{dx} > 0$ when $xy > 0$. Quadrants I and III.

28. $\frac{dy}{dx} = \frac{1}{2}x^2y$

$\frac{dy}{dx} > 0$ when $y > 0$. Quadrants I and II.

29. Because the initial quantity is 20 grams,

$y = 20e^{kt}$.

Because the half-life is 1599 years,

$10 = 20e^{k(1599)}$

$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$.

So, $y = 20e^{[\ln(1/2)/1599]t}$.

When $t = 1000$, $y = 20e^{[\ln(1/2)/1599](1000)} \approx 12.96g$.

When $t = 10{,}000$, $y \approx 0.26g$.

30. Because the half-life is 1599 years,

$\frac{1}{2} = 1e^{k(1599)}$

$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$.

Because there are 1.5 g after 1000 years,

$1.5 = Ce^{[\ln(1/2)/1599](1000)}$

$C \approx 2.314$.

So, the initial quantity is approximately 2.314 g.

When $t = 10{,}000$, $y = 2.314e^{[\ln(1/2)/1599](10{,}000)}$

≈ 0.03 g.

31. Because the half-life is 1599 years,

$\frac{1}{2} = 1e^{k(1599)}$

$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$.

Because there are 0.1 gram after 10,000 years,

$0.1 = Ce^{[\ln(1/2)/1599](10{,}000)}$

$C \approx 7.63$.

So, the initial quantity is approximately 7.63 g.

When $t = 1000$, $y = 7.63e^{[\ln(1/2)/1599](1000)}$

≈ 4.95 g.

32. Because the half-life is 5715 years,

$\frac{1}{2} = 1e^{k(5715)}$

$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right)$.

Because there are 3 grams after 10,000 years,

$3 = Ce^{[\ln(1/2)/5715](10{,}000)}$

$C \approx 10.089$.

So, the initial quantity is approximately 10.09 g.

When $t = 1000$, $y = 10.09e^{[\ln(1/2)/5715](1000)}$

≈ 8.94 g.

33. Because the initial quantity is 5 grams, $C = 5$.

Because the half-life is 5715 years,

$2.5 = 5e^{k(5715)}$

$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right)$.

When $t = 1000$ years, $y = 5e^{[\ln(1/2)/5715](1000)} \approx 4.43$ g.

When $t = 10{,}000$ years, $y = 5e^{[\ln(1/2)/5715](10{,}000)}$

≈ 1.49 g.

34. Because the half-life is 5715 years,

$\frac{1}{2} = 1e^{k(5715)}$

$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right).$

Because there are 1.6 grams when $t = 1000$ years,

$1.6 = Ce^{[\ln(1/2)/5715](1000)}$

$C \approx 1.806.$

So, the initial quantity is approximately 1.806 g.

When $t = 10{,}000$, $y = 1.806e^{[\ln(1/2)/5715](10{,}000)}$

≈ 0.54 g.

35. Because the half-life is 24,100 years,

$\frac{1}{2} = 1e^{k(24{,}100)}$

$k = \frac{1}{24{,}100}\ln\left(\frac{1}{2}\right).$

Because there are 2.1 grams after 1000 years,

$2.1 = Ce^{[\ln(1/2)/24{,}100](1000)}$

$C \approx 2.161.$

So, the initial quantity is approximately 2.161 g.

When $t = 10{,}000$, $y = 2.161e^{[\ln(1/2)/24{,}100](10{,}000)}$

≈ 1.62 g.

36. Because the half-life is 24,100 years,

$\frac{1}{2} = 1e^{k(24{,}100)}$

$k = \frac{1}{24{,}100}\ln\left(\frac{1}{2}\right).$

Because there are 0.4 grams after 10,000 years,

$0.4 = Ce^{[\ln(1/2)/24{,}100](10{,}000)}$

$C \approx 0.533.$

So, the initial quantity is approximately 0.533 g.

When $t = 1000$, $y = 0.533e^{[\ln(1/2)/24{,}100](1000)}$

≈ 0.52 g.

37. $y = Ce^{kt}$

$\frac{1}{2}C = Ce^{k(1599)}$

$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$

When $t = 100$, $y = Ce^{[\ln(1/2)/1599](100)}$

$\approx 0.9576\text{ C}$

Therefore, 95.76% remains after 100 years.

38. $y = Ce^{kt}$

$\frac{1}{2}C = Ce^{k(5715)}$

$k = \frac{1}{5715}\ln\left(\frac{1}{2}\right)$

$0.15C = Ce^{[\ln(1/2)/5715]t}$

$\ln(0.15) = \frac{\ln\left(\frac{1}{2}\right)t}{5715}$

$t \approx 15{,}641.8$ years

39. Because $A = 4000e^{0.06t}$, the time to double is given by

$8000 = 4000e^{0.06t}$

$2 = e^{0.06t}$

$\ln 2 = 0.06t$

$t = \frac{\ln 2}{0.06} \approx 11.55$ years.

Amount after 10 years: $A = 4000e^{(0.06)(10)} \approx \$7288.48$

40. Because $A = 18{,}000e^{0.055t}$, the time to double is given by

$36{,}000 = 18{,}000e^{0.055t}$

$2 = e^{0.055t}$

$\ln 2 = 0.055t$

$t = \frac{\ln 2}{0.055} \approx 12.6$ years.

Amount after 10 years:
$A = 18{,}000e^{(0.055)(10)} \approx \$31{,}198.55$

41. Because $A = 750e^{rt}$ and $A = 1500$ when $t = 7.75$, you have the following.

$1500 = 750e^{7.75r}$

$2 = e^{7.75r}$

$\ln 2 = 7.75r$

$r = \frac{\ln 2}{7.75} \approx 0.0894 = 8.94\%$

Amount after 10 years: $A = 750e^{0.0894(10)} \approx \$1833.67$

42. Because $A = 12{,}500e^{rt}$ and $A = 25{,}000$ when $t = 20$, you have the following.

$25{,}000 = 12{,}500e^{20r}$

$2 = e^{20r}$

$\ln 2 = 20r$

$r = \frac{\ln 2}{20} \approx 0.03466 \approx 3.47\%$

Amount after 10 years:
$A = 12{,}500e^{0.03466(10)} \approx \$17{,}678.14$

43. Because $A = 500e^{rt}$ and $A = 1292.85$ when $t = 10$, you have the following.

$$1292.85 = 500e^{10r}$$
$$2.5857 = e^{10r}$$
$$\ln(2.5857) = 10r$$
$$r = \frac{\ln(2.5857)}{10} \approx 0.0950 = 9.50\%$$

The time to double is given by

$$1000 = 500e^{0.0950t}$$
$$2 = e^{0.0950t}$$
$$\ln 2 = 0.0950t$$
$$t = \frac{\ln 2}{0.095} \approx 7.30 \text{ years.}$$

44. Because $A = 6000e^{rt}$ and $A = 8950.95$ when $t = 10$, you have the following.

$$8950.95 = 6000e^{10r}$$
$$\frac{8950.95}{6000} = e^{10r}$$
$$\ln\left(\frac{8950.95}{6000}\right) = 10r$$
$$r = \frac{1}{10}\ln\frac{8950.95}{6000} = 0.04 = 4\%$$

The time to double is given by

$$12{,}000 = 6000e^{0.04t}$$
$$2 = e^{0.04t}$$
$$\ln 2 = 0.04t$$
$$t = \frac{\ln 2}{0.04} \approx 17.33 \text{ years.}$$

45. $$1{,}000{,}000 = P\left(1 + \frac{0.075}{12}\right)^{(12)(20)}$$
$$P = 1{,}000{,}000\left(1 + \frac{0.075}{12}\right)^{-240}$$
$$\approx \$224{,}174.18$$

46. $$1{,}000{,}000 = P\left(1 + \frac{0.06}{12}\right)^{(12)(40)}$$
$$P = 1{,}000{,}000(1.005)^{-480} \approx \$91{,}262.08$$

47. $$1{,}000{,}000 = P\left(1 + \frac{0.08}{12}\right)^{(12)(35)}$$
$$P = 1{,}000{,}000\left(1 + \frac{0.08}{12}\right)^{-420}$$
$$= \$61{,}377.75$$

48. $$1{,}000{,}000 = P\left(1 + \frac{0.09}{12}\right)^{(12)(25)}$$
$$P = 1{,}000{,}000\left(1 + \frac{0.09}{12}\right)^{-300}$$
$$\approx \$106{,}287.83$$

49. (a) $$2000 = 1000(1 + 0.07)^t$$
$$2 = 1.07^t$$
$$\ln 2 = t \ln 1.07$$
$$t = \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ years}$$

(b) $$2000 = 1000\left(1 + \frac{0.07}{12}\right)^{12t}$$
$$2 = \left(1 + \frac{0.007}{12}\right)^{12t}$$
$$\ln 2 = 12t \ln\left(1 + \frac{0.07}{12}\right)$$
$$t = \frac{\ln 2}{12\ln(1 + (0.07/12))} \approx 9.93 \text{ years}$$

(c) $$2000 = 1000\left(1 + \frac{0.07}{365}\right)^{365t}$$
$$2 = \left(1 + \frac{0.07}{365}\right)^{365t}$$
$$\ln 2 = 365t \ln\left(1 + \frac{0.07}{365}\right)$$
$$t = \frac{\ln 2}{365\ln(1 + (0.07/365))} \approx 9.90 \text{ years}$$

(d) $$2000 = 1000e^{(0.07)t}$$
$$2 = e^{0.07t}$$
$$\ln 2 = 0.07t$$
$$t = \frac{\ln 2}{0.07} \approx 9.90 \text{ years}$$

50. (a) $2000 = 1000(1 + 0.055)^t$

$2 = 1.055^t$

$\ln 2 = t \ln 1.055$

$t = \dfrac{\ln 2}{\ln 1.055} \approx 12.95$ years

(b) $2000 = 1000\left(1 + \dfrac{0.055}{12}\right)^{12t}$

$2 = \left(1 + \dfrac{0.055}{12}\right)^{12t}$

$\ln 2 = 12t \ln\left(1 + \dfrac{0.055}{12}\right)$

$t = \dfrac{1}{12}\dfrac{\ln 2}{\ln\left(1 + \dfrac{0.055}{12}\right)} \approx 12.63$ years

(c) $2000 = 1000\left(1 + \dfrac{0.055}{365}\right)^{365t}$

$2 = \left(1 + \dfrac{0.055}{365}\right)^{365t}$

$\ln 2 = 365t \ln\left(1 + \dfrac{0.055}{365}\right)$

$t = \dfrac{1}{365}\dfrac{\ln 2}{\ln\left(1 + \dfrac{0.055}{365}\right)} \approx 12.60$ years

(d) $2000 = 1000e^{0.055t}$

$2 = e^{0.055t}$

$\ln 2 = 0.055t$

$t = \dfrac{\ln 2}{0.055} \approx 12.60$ years

51. (a) $P = Ce^{kt} = Ce^{-0.006t}$

$P(1) = 2.2 = Ce^{-0.006(1)} \Rightarrow C \approx 2.21$

$P = 2.21e^{-0.006t}$

(b) For 2020, $t = 10$ and

$P = 2.21e^{-0.006(10)} \approx 2.08$ million.

(c) Because $k < 0$, the population is decreasing.

52. (a) $P = Ce^{kt} = Ce^{0.020t}$

$P(1) = 82.1 = Ce^{0.020(1)} \Rightarrow C \approx 80.47$

$P = 80.47e^{0.020t}$

(b) For 2020, $t = 10$ and

$P = 80.47e^{0.020(10)} \approx 98.29$ million.

(c) Because $k > 0$, the population is increasing.

53. (a) $P = Ce^{kt} = Ce^{0.036t}$

$P(1) = 34.6 = Ce^{0.036(1)} \Rightarrow C \approx 33.38$

$P = 33.38e^{0.036t}$

(b) For 2020, $t = 10$ and

$P = 33.38e^{0.036(10)} \approx 47.84$ million.

(c) Because $k > 0$, the population is increasing.

54. (a) $P = Ce^{kt} = Ce^{-0.002t}$

$P(1) = 10.0 = Ce^{-0.002(1)} \Rightarrow C \approx 10.02$

$P = 10.02e^{-0.002t}$

(b) For 2020, $t = 10$ and

$P = 10.02e^{-0.002(10)} \approx 9.82$ million.

(c) Because $k < 0$, the population is decreasing.

55. (a) $N = 100.1596(1.2455)^t$

(b) $N = 400$ when $t = 6.3$ hours (graphing utility)

Analytically,

$400 = 100.1596(1.2455)^t$

$1.2455^t = \dfrac{400}{100.1596} = 3.9936$

$t \ln 1.2455 = \ln 3.9936$

$t = \dfrac{\ln 3.9936}{\ln 1.2455} \approx 6.3$ hours

56. (a) Let $y = Ce^{kt}$.

At time 2: $125 = Ce^{k(2)} \Rightarrow C = 125e^{-2k}$

At time 4:

$350 = Ce^{k(4)} \Rightarrow 350 = \left(125e^{-2k}\right)\left(e^{4k}\right)$

$\frac{14}{5} = e^{2k}$

$2k = \ln \frac{14}{5}$

$k = \frac{1}{2} \ln \frac{14}{5} \approx 0.5148$

$C = 125e^{-2k}$

$= 125e^{-2(1/2)\ln(14/5)}$

$= 125\left(\frac{5}{14}\right) = \frac{625}{14} \approx 44.64$

Approximately 45 bacteria at time 0.

(b) $y = \frac{625}{14} e^{(1/2)\ln(14/5)t} \approx 44.64e^{0.5148t}$

(c) When $t = 8$,

$y = \frac{625}{14}e^{(1/2)\ln(14/5)8} = \frac{625}{14}\left(\frac{14}{5}\right)^4 = 2744.$

(d) $25{,}000 = \frac{625}{14}e^{(1/2)\ln(14/5)t} \Rightarrow t \approx 12.29$ hours

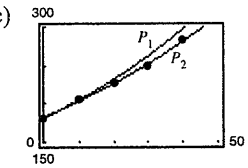
57. (a) $P_1 = Ce^{kt} = 181e^{kt}$

$$205 = 181e^{10k} \Rightarrow k = \frac{1}{10}\ln\left(\frac{205}{181}\right) \approx 0.01245$$

$$P_1 \approx 181e^{0.01245t} \approx 181(1.01253)^t$$

(b) Using a graphing utility, $P_2 \approx 182.3248(1.01091)^t$

(c)



The model P_2 fits the data better.

(d) Using the model P_2,

$$320 = 182.3248(1.01091)^t$$

$$\frac{320}{182.3248} = (1.01091)^t$$

$$t = \frac{\ln(320/182.3248)}{\ln(1.01091)}$$

$$\approx 51.8 \text{ years, or } 2011.$$

58. (a) $20 = 30(1 - e^{30k})$

$$30e^{30k} = 10$$

$$k = \frac{\ln(1/3)}{30} = \frac{-\ln 3}{30} \approx -0.0366$$

$$N \approx 30(1 - e^{-0.0366t})$$

(b) $25 = 30(1 - e^{-0.0366t})$

$$e^{-0.0366t} = \frac{1}{6}$$

$$t = \frac{-\ln 6}{-0.0366} \approx 49 \text{ days}$$

59. (a) Because the population increases by a constant each month, the rate of change from month to month will always be the same. So, the slope is constant, and the model is linear.

(b) Although the percentage increase is constant each month, the rate of growth is not constant. The rate of change of y is given by

$$\frac{dy}{dt} = ry$$

which is an exponential model.

60. (a) Both functions represent exponential growth because the graphs are increasing.

(b) g has a greater k value because its graph is increasing at a greater rate than the graph of f.

61. (a) $P_1 = Ce^{kt} = 106e^{kt}$ $(t = 0 \leftrightarrow 1920)$

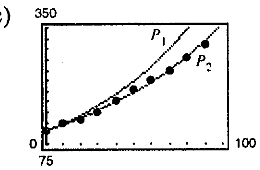
$$123 = 106e^{k(10)} \Rightarrow \frac{123}{106} = e^{10k}$$

$$\Rightarrow k = \frac{1}{10}\ln\left(\frac{123}{106}\right) \approx 0.01487$$

$$P_1 = 106e^{0.01487t} = 106e^{\frac{1}{10}\ln\left(\frac{123}{106}\right)t} = 106(1.01499)^t$$

(b) Using a graphing utility, $P_2 \approx 107.2727(1.01215)^t$.

(c)



The model P_2 fits the data better.

(d) $P_2 = 400 = 107.2727(1.01215)^t$

$$\frac{400}{107.2727} = (1.01215)^t$$

$$t = \frac{\ln(400/107.2727)}{\ln(1.01215)}$$

$$\approx 109, \text{ or } 2029.$$

62. $A(t) = V(t)e^{-0.10t}$

$$= 100{,}000e^{0.8\sqrt{t}}e^{-0.10t} = 100{,}000e^{0.8\sqrt{t}-0.10t}$$

$$\frac{dA}{dt} = 100{,}000\left(\frac{0.4}{\sqrt{t}} - 0.10\right)e^{0.8\sqrt{t}-0.10t}$$

$$\frac{dA}{dt} = 0 \text{ when } \frac{0.4}{\sqrt{t}} = 0.10 \Rightarrow t = 16.$$

The timber should be harvested in the year 2026 $(2010 + 16)$.

Note: You could also use a graphing utility to graph $A(t)$ and find the maximum value. Use a viewing window of $0 \le x \le 30$, $0 \le y \le 600{,}000$.

63. $\beta(I) = 10\log_{10}\frac{I}{I_0}$, $I_0 = 10^{-16}$

(a) $\beta(10^{-14}) = 10\log_{10}\frac{10^{-14}}{10^{-16}} = 20$ decibels

(b) $\beta(10^{-9}) = 10\log_{10}\frac{10^{-9}}{10^{-16}} = 70$ decibels

(c) $\beta(10^{-6.5}) = 10\log_{10}\frac{10^{-6.5}}{10^{-16}} = 95$ decibels

(d) $\beta(10^{-4}) = 10\log_{10}\frac{10^{-4}}{10^{-16}} = 120$ decibels

64. $93 = 10\log_{10}\dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-6.7 = \log_{10} I \Rightarrow I = 10^{-6.7}$

$80 = 10\log_{10}\dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-8 = \log_{10} I \Rightarrow I = 10^{-8}$

Percentage decrease: $\left(\dfrac{10^{-6.7} - 10^{-8}}{10^{-6.7}}\right)(100) \approx 95\%$

65. Because $\dfrac{dy}{dt} = k(y - 80)$

$\int \dfrac{1}{y - 80}\,dy = \int k\,dt$

$\ln(y - 80) = kt + C.$

When $t = 0, y = 1500$. So, $C = \ln 1420$.

When $t = 1, y = 1120$. So,

$k(1) + \ln 1420 = \ln(1120 - 80)$

$k = \ln 1040 - \ln 1420 = \ln \dfrac{104}{142}.$

So, $y = 1420e^{[\ln(104/142)]t} + 80.$

When $t = 5, y \approx 379.2°\text{F}$.

66. $\dfrac{dy}{dt} = k(y - 20)$

$y = 20 + Ce^{kt}$ (See Example 6.)

$160 = 20 + Ce^{k(0)} \Rightarrow C = 140$

$60 = 20 + 140e^{k(5)}$

$\dfrac{2}{7} = e^{5k}$

$k = \dfrac{1}{5}\ln\left(\dfrac{2}{7}\right) \approx -0.25055$

$30 = 20 + 140e^{(1/5)\ln(2/7)t}$

$\dfrac{1}{14} = e^{\ln(2/7)^{t/5}} = \left(\dfrac{2}{7}\right)^{t/5}$

$\ln\dfrac{1}{14} = \dfrac{t}{5}\ln\dfrac{2}{7}$

$t = \dfrac{5\ln\dfrac{1}{14}}{\ln\dfrac{2}{7}} = \dfrac{5\ln 14}{\ln\dfrac{7}{2}} \approx 10.53$ minutes

It will take $10.53 - 5 = 5.53$ minutes longer.

67. False. If $y = Ce^{kt}$, $y' = Cke^{kt} \neq$ constant.

68. True

69. False. The prices are rising at a rate of 6.2% per year.

70. True

Section 6.3 Separation of Variables and the Logistic Equation

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$

$\int y\,dy = \int x\,dx$

$\dfrac{y^2}{2} = \dfrac{x^2}{2} + C_1$

$y^2 - x^2 = C$

2. $\dfrac{dy}{dx} = \dfrac{3x^2}{y^2}$

$\int y^2\,dy = \int 3x^2\,dx$

$\dfrac{y^3}{3} = x^3 + C_1$

$y^3 - 3x^3 = C$

3. $x^2 + 5y\dfrac{dy}{dx} = 0$

$5y\dfrac{dy}{dx} = -x^2$

$\int 5y\,dy = \int -x^2\,dx$

$\dfrac{5y^2}{2} = \dfrac{-x^3}{3} + C_1$

$15y^2 + 2x^3 = C$

4. $\dfrac{dy}{dx} = \dfrac{6 - x^2}{2y^3}$

$\int 2y^3\,dy = \int (6 - x^2)\,dx$

$\dfrac{y^4}{2} = 6x - \dfrac{x^3}{3} + C_1$

$3y^4 + 2x^3 - 36x = C$