AP Calculus – 6.2b Notes –Summation Notation and Limit Definition of Area

Key

I. Sigma Notation

$$\sum_{i=2}^{5} a_i = a_2 + a_3 + a_4 + a_5$$

upper bound

index

lower bound

argument expression (formula Rule)

Ex. 1 $\sum_{i=2}^{4} i^2+1 = (2^2+1)+(3^2+1)+(4^2+1) = 5+10+17 = \boxed{32}$

II. Summation Formulas:

1) $\sum_{i=1}^{n} 1 = n$

2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

5) $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

Example 2

$$\sum_{i=1}^{8}(3i^2+2) =$$

$$3\sum_{i=1}^{8} i^2 + \sum_{i=1}^{8} 2$$

$$\frac{3(8)(8+1)(16+1)}{6} + 2(8) = 612+16 = \boxed{628}$$

Example 3

$$\sum_{i=1}^{10}(i+2)^2 = (i+2)(i+2)$$

$$\sum i^2+4i+4$$

$$\sum_{i=1}^{10} i^2 + 4\sum_{i=1}^{10} i + 4\sum_{i=1}^{10} 1$$

$$385 + 220 + 40 = \boxed{645}$$

Example 4

$$\sum_{k=1}^{n}\frac{1}{n}(k^2-1) = \sum \frac{1}{n}k^2 - \frac{1}{n}(1)$$

$$\frac{1}{n}\sum_{k=1}^{n} k^2 - \frac{1}{n}\sum_{k=1}^{n} 1$$

$$\frac{1}{n}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{1}{n}(n)$$

$$\frac{(n+1)(2n+1)}{6} - 1 \rightarrow \frac{2n^2+3n+1}{6} - \frac{6}{6} = \boxed{\frac{2n^2+3n-5}{6}}$$

III. Limits as n approaches infinity

***Think back about finding horizontal asymptotes**

(*compare degrees between numerator and denominator)

Example 5: *If* $S(n) = \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$ *, then find* $\lim_{n\to\infty} S(n)$

$$\lim_{n\to\infty} \frac{1}{n^2}\cdot\frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} \frac{n^2+n}{2n^2} = \frac{1}{2}$$

Example 6: *Find* $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4i}{n^2}$$

$$\lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n} i$$

$$\lim_{n\to\infty}\frac{4}{n^2}\cdot\frac{n(n+1)}{2}$$

$$\lim_{n\to\infty}\frac{4n^2+4n}{2n^2}=\frac{4}{2}=\boxed{2}$$

Example 7: *Find* $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^2\left(\frac{2}{n}\right) \to \frac{2}{n}\left(1+\frac{2i}{n}\right)\left(1+\frac{2i}{n}\right)$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(1+\frac{4i}{n}+\frac{4i^2}{n^2}\right)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}+\frac{8i}{n^2}+\frac{8i^2}{n^3}$$

$$\lim_{n\to\infty}\left[\frac{2}{n}\sum_{i=1}^{n}1+\frac{8}{n^2}\sum_{i=1}^{n}i+\frac{8}{n^3}\sum_{i=1}^{n}i^2\right]$$

$$\lim_{n\to\infty}\frac{2}{n}(n)+\frac{8}{n^2}\left(\frac{n(n+1)}{2}\right)+\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

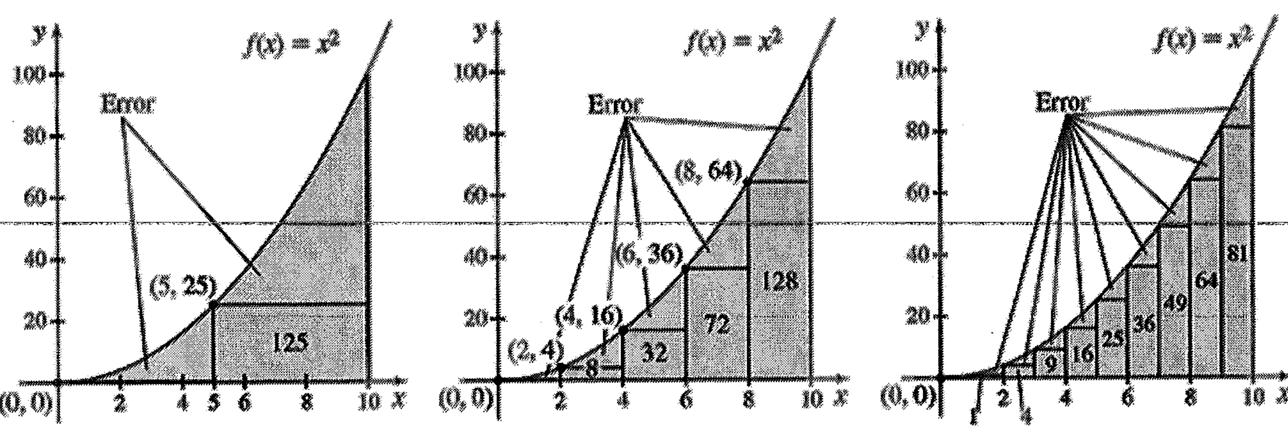
$$\lim_{n\to\infty}\frac{2n}{n}+\frac{8n^2}{2n^2}+\frac{16n^3}{6n^3}$$

$$2+4+\frac{8}{3}=\boxed{\frac{26}{3}}$$

Moving from Area Approximation to Exact Area calculations:

We can continually improve the Area Approximation under the curve by increasing the number of rectangles (n = 2 to n = 5 to n = 10). If we let ***n*** go out to infinity (using limits), we will have something better than an approximation, we will achieve the actual area under the curve:

(a) $n = 2$ **subintervals** **(b)** $n = 5$ **subintervals** **(c)** $n = 10$ **subintervals**



(a) Two subintervals (b) Five subintervals (c) Ten subintervals

If $n \to \infty$ **on the interval** $[a, b]$**, what does the width of each subinterval (rectangle) approach?** As $n\to\infty$, rectangle width approaches zero.

Finding Area Using Limit Definition:

$$\overbrace{\text{width}}\cdot f[\text{leftendpt} + \text{width}\cdot i]$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{b-a}{n}\right)*f\left(a+\frac{b-a}{n}*i\right)$$

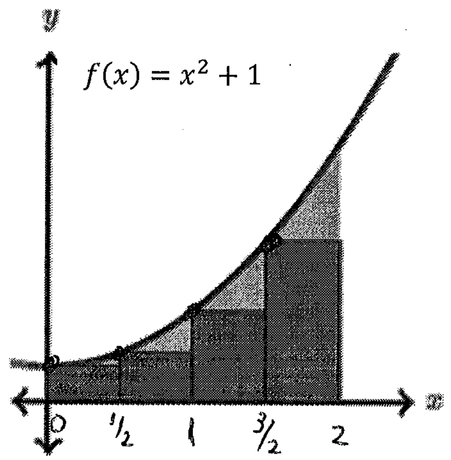
$$\boxed{\text{width} = \frac{b-a}{n}}$$

Definite Integral Notation:

The area under the curve of $f(x)$ on the interval $[a,b]$ is represented by $\int_a^b f(x)\,dx$.

8a) Approximate the area under the curve $f(x) = x^2 + 1$ using 4 rectangles on interval [0,2]

$$W = \frac{2-0}{4} = \frac{1}{2}$$



$$\int_0^2 x^2+1\,dx \approx \frac{1}{2}f(0) + \frac{1}{2}\cdot f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{3}{2}\right)$$

$$\approx \frac{1}{2}(1) + \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right)$$

$$= \boxed{3.75}$$

$f(0) = 1$ | $f(1) = 2$

$f(1/2) = 5/4$ | $f(3/2) = 13/4$

b) Find the exact area under the curve $f(x) = x^2 + 1$ using limit definition of area on the interval [0,2]

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2-0}{n}\right)\cdot f\left[0+\frac{2}{n}i\right]$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\cdot f\left(\frac{2}{n}i\right)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\cdot\left[\left(\frac{2}{n}i\right)^2+1\right]$$

$$\lim_{n}\frac{2}{n}\left[\frac{4}{n^2}i^2+1\right]$$

$$f\left(\frac{2}{n}i\right) = \left(\frac{2}{n}i\right)^2+1$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{8}{n^3}i^2+\frac{2}{n}\right]$$

$$\lim_{n\to\infty}\frac{8}{n^3}\sum_{i=1}^{n}i^2+\frac{2}{n}\sum_{i=1}^{n}1$$

$$\lim_{n\to\infty}\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{2}{n}\cdot n$$

$$\lim_{n\to\infty}\frac{16n^3+\dots}{6n^3}+\frac{2n}{n}$$

$$\frac{16}{6}+2 = \boxed{\frac{14}{3}}$$

Practice problems:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{b-a}{n}\right)*f\left(a+\frac{b-a}{n}*i\right)$$

$W=\frac{6-2}{n}=\frac{4}{n}$ width $=\frac{b-a}{n}$ $f(x)=$

$f\left(2+\frac{4}{n}i\right)$ $f(\ )=(\ )^2-3$

$f\left(2+\frac{4}{n}i\right)=\left(2+\frac{4}{n}i\right)^2-3$

Examples:

1. **Rewrite the definite integral using summation notation.**

$$\int_2^6 (x^2-3)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{4}{n}\right)\left[\left(2+\frac{4}{n}i\right)^2-3\right]$$

2. **Rewrite the summation notation expression as a definite integral.**

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{6}{n}\right)\left(4+\frac{6k}{n}\right)^2=$$

a. $\int_0^6 (4+x)^2\,dx$

$W=\frac{6-0}{n}=\frac{6}{n}$

$\left(\frac{6}{n}\right)\left[4+\frac{6}{n}k\right]^2$

$f\left(a+\frac{6}{n}i\right)$

$f\left(0+\frac{6}{n}i\right)$

$f(x)=(4+x)^2$

b. $\int_4^{10} x^2\,dx$

$W=\frac{10-4}{n}=\frac{6}{n}$

$f\left(a+\frac{6}{n}i\right)$

$f\left(4+\frac{6}{n}i\right)$

$f(x)=x^2$

c. $\int_0^1 6(4+6x)^2\,dx$

$W=\frac{1-0}{n}=\frac{1}{n}$

$f\left(a+\frac{1}{n}i\right)$

$f\left(0+\frac{1}{n}i\right)$

$\frac{1}{n}\cdot 6\left(4+6\left(\frac{1}{n}i\right)\right)^2$

$\frac{6}{n}\left[4+\frac{6}{n}k\right]^2$

3. $$\lim_{n\to\infty}\frac{2}{n}\left(\frac{1}{\frac{2}{n}+3}+\frac{1}{\frac{4}{n}+3}+\frac{1}{\frac{6}{n}+3}+\cdots+\frac{1}{\frac{2n}{n}+3}\right)$$

Assuming the lower limit "a" is 0, write a definite integral that represents the above expression.

$\frac{b-a}{n}\to\frac{2-0}{n}$

$f(x)=\frac{1}{x+3}$

$f\left(a+\frac{2}{n}i\right)$

$f\left(0+\frac{2}{n}i\right)$

$$\int_0^2\left(\frac{1}{x+3}\right)dx$$

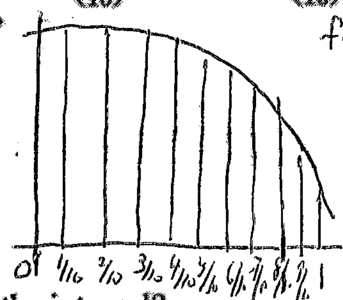
4. **The expression** $\frac{1}{10}\left(\cos\left(\frac{1}{10}\right)+\cos\left(\frac{2}{10}\right)+\cos\left(\frac{3}{10}\right)+\cdots+\cos\left(\frac{10}{10}\right)\right)$ **is a Riemann sum approximation for what definite integral?**

$f(x)=\cos x$

10 rectangles to approximate this area

$\frac{1}{10}f\left(\frac{1}{10}\right)+\frac{1}{10}\cdot f\left(\frac{2}{10}\right)+\frac{1}{10}f\left(\frac{3}{10}\right)+\cdots$

$\frac{b-a}{n}\to\frac{1-0}{10}=\frac{1}{10}$



Where is the 10? Why isn't it written in the integral?