6.3 Homogeneous Differential Equations: p. 429 #29–40 all

29) $f(x,y) = \dfrac{x^2y^2}{\sqrt{x^2+y^2}}$ $\quad f(tx,ty) = \dfrac{(tx)^2(ty)^2}{\sqrt{(tx)^2+(ty)^2}} = \dfrac{t^4(x^2y^2)}{\sqrt{t^2(x^2+y^2)}}$

$= \dfrac{t^4(x^2y^2)}{t\sqrt{x^2+y^2}} = t^3\left(\dfrac{x^2y^2}{\sqrt{x^2+y^2}}\right)$ $\boxed{\text{Homogeneous of degree 3}}$

30) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$ $\quad f(tx,ty) = \dfrac{(tx)(ty)}{\sqrt{(tx)^2+(ty)^2}} = \dfrac{t^2xy}{\sqrt{t^2(x^2+y^2)}}$

$\boxed{\text{Homogeneous of degree 1.}}$ $\quad = \dfrac{t^2}{t}\left[\dfrac{xy}{\sqrt{x^2+y^2}}\right] = t\left[\dfrac{xy}{\sqrt{x^2+y^2}}\right]$

31) $f(x,y) = 2\ln xy$ $\quad f(tx,ty) = 2\ln(tx)(ty) = 2\ln(t^2xy)$

$= 2\ln(t^2) + 2\ln(xy)$

Not homogeneous

32) $f(x,y) = \tan(x+y)$

$f(tx,ty) = \tan(tx+ty) = \tan[t(x+y)]$ Not homogeneous.

33) $f(x,y) = 2\ln\dfrac{x}{y}$

$f(tx,ty) = 2\ln\dfrac{tx}{ty} = 1\left[2\ln\dfrac{x}{y}\right]$ homogeneous degree 0

$= 1t^0\left[2\ln\dfrac{x}{y}\right]$

34) $f(x,y) = \tan\dfrac{y}{x}$

$f(tx,ty) = \tan\dfrac{tx}{ty} = \tan\dfrac{x}{y}$

$\boxed{\text{homogeneous of degree 0.}}$

35) Solve homogeneous differential equation

$$y' = \frac{x+y}{2x} \qquad \frac{dy}{dx} = \frac{x+y \cdot \frac{1}{x}}{2x \cdot \frac{1}{x}} = \frac{1+\frac{y}{x}}{2} = \frac{1+v}{2}$$

Degree: $n=1$

$$v = \frac{y}{x} \qquad xv = y \qquad y = xv \qquad \frac{dy}{dx} = 1v + x\left(\frac{dv}{dx}\right)$$

$$v + x\left(\frac{dv}{dx}\right) = \frac{1+v}{2}$$

$$x\frac{dv}{dx} = \frac{1+v}{2} - v$$

$$x\frac{dv}{dx} = \frac{1+v}{2} - \frac{2v}{2} = \frac{1-v}{2}$$

$$x\frac{dv}{dx} = \frac{1-v}{2}$$

$$\int \frac{dv}{1-v} = \int \frac{1}{2x}\,dx \rightarrow \frac{1}{2}\int \frac{1}{x}\,dx$$

$u = 1-v$

$\frac{du}{dv} = -1$

$dv = \frac{du}{-1}$

$$\int \frac{-du}{u} = \frac{1}{2}\ln|x| + C$$

$$\ln|1-v|^{-1} = \frac{1}{2}\ln|x| + C$$

$$e^{\ln|1-v|^{-1}} = e^{\ln|x|^{1/2} + C}$$

$$|1-v|^{-1} = x^{1/2} \cdot e^{C}$$

$$|1-v|^{-1} = Cx^{1/2}$$

$$\frac{1}{1-v} = Cx^{1/2}$$

$$\frac{1}{1-\frac{y}{x}} = Cx^{1/2}$$

$$\left(\frac{1}{1-\frac{y}{x}}\right)^2 = \left(Cx^{1/2}\right)^2$$

$$\frac{1}{\left(1-\frac{y}{x}\right)^2} = Cx$$

$$\frac{1}{\left(\frac{x-y}{x}\right)^2} = Cx$$

$$\frac{x^2}{(x-y)^2} = Cx$$

$$\frac{x^2}{x} = C(x-y)^2$$

$$\boxed{x = C(x-y)^2}$$

36) $y' = \dfrac{(x^3+y^3)}{xy^2}$

homogeneous : degree $n=3$

$$\frac{dy}{dx} = \frac{(x^3+y^3)\cdot\frac{1}{x^3}}{xy^2\cdot\frac{1}{x^3}} = \frac{1+\frac{y^3}{x^3}}{\frac{y^2}{x^2}} = \frac{1+\left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^2}$$

$V = \frac{y}{x}$

$xv = y$

$y = xv$

$\frac{dy}{dx} = 1v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{1+v^3}{v^2}$

$x\frac{dv}{dx} = \frac{1+v^3}{v^2} - v$

$x\frac{dv}{dx} = \frac{1+v^3-v^3}{v^2}$

$x\frac{dv}{dx} = \frac{1}{v^2}$

$\int v^2 dv = \int \frac{1}{x} dx$

$\int v^2 dv = \int \frac{1}{x} dx$

$\frac{v^3}{3} = \ln|x| + c$

$v^3 = 3\ln|x| + c$

$v^3 = 3\ln|x| + c$

$\left(\frac{y}{x}\right)^3 = 3\ln|x| + c$

$\frac{y^3}{x^3} = \ln|x| + c$

$\boxed{y^3 = 3x^3\ln|x| + Cx^3}$

37) $y' = \dfrac{x-y}{x+y}$

homogeneous : degree $n=1$

$$\frac{dy}{dx} = \frac{x-y}{x+y}\cdot\frac{\cdot\frac{1}{x}}{\cdot\frac{1}{x}} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$$

$\frac{dy}{dx} = 1v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{1-v}{1+v}$

$x\frac{dv}{dx} = \frac{1-v}{1+v} - v$

$x\frac{dv}{dx} = \frac{1-v-v(1+v)}{1+v}$

$x\frac{dv}{dx} = \frac{1-v-v-v^2}{1+v}$

$= \frac{1-2v-v^2}{1+v}$

$x\frac{dv}{dx} = \frac{-v^2-2v+1}{1+v}$

$\int \frac{1+v}{-v^2-2v+1} dv = \int \frac{1}{x} dx$

$\int \frac{1+v}{-v^2-2v+1} = \int \frac{1}{x} dx$

$u = -v^2-2v+1$

$\frac{du}{dv} = -2v-2$

$\frac{du}{dv} = -2(v+1)$

$dv = \frac{du}{-2(v+1)}$

$\int \frac{1+\cancel{v}}{u}\cdot\frac{du}{-2\cancel{(1+v)}}$

$-\frac{1}{2}\ln|-v^2-2v+1| = \ln|x| + C$

$e^{\ln|-v^2-2v+1|^{-1/2}} = e^{\ln|x|+C}$

$\left(|1-v^2-2v+1|^{-1/2}\right)^{-2} = (C|x|)^{-2}$

$|1-v^2-2v+1| = Cx^{-2}$

$\left|1-\left(\frac{y}{x}\right)^2-2\left(\frac{y}{x}\right)+1\right| = \frac{C}{x^2}$

$\boxed{|-y^2-2xy-x^2| = C}$

38) $y' = \dfrac{x^2+y^2}{2xy}$ homogeneous degree: $n=2$

$\dfrac{dy}{dx} = \dfrac{x^2+y^2}{2xy} \cdot \dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$\dfrac{dy}{dx} = \dfrac{1+\frac{y^2}{x^2}}{2\left(\frac{y}{x}\right)}$

$\dfrac{y}{x} = v \quad y = vx$

$\dfrac{dy}{dx} = \dfrac{dv}{dx}x + v(1)$

$x\dfrac{dv}{dx} + v = \dfrac{1+v^2}{2v}$

$x\dfrac{dv}{dx} = \dfrac{1+v^2}{2v} - v$

$x\dfrac{dv}{dx} = \dfrac{1+v^2-2v^2}{2v}$

$x\dfrac{dv}{dx} = \dfrac{1-v^2}{2v}$

$\dfrac{2v\,dv}{1-v^2} = \dfrac{1}{x}dx$

$\displaystyle\int \frac{2v}{1-v^2}\,dv = \int \frac{1}{x}\,dx$

$u = 1-v^2$

$\dfrac{du}{dv} = -2v$

$dv = \dfrac{du}{-2v}$

$\displaystyle\int \frac{2v}{u} \cdot \frac{du}{-2v}$

$-\ln|1-v^2| = \ln|x| + c$

$e^{\ln|1-v^2|^{-1}} = e^{\ln|x|+c}$

$|1-v^2|^{-1} = C|x|$

$|1-v^2|^{-1} = C|x|$

$1-v^2 = \dfrac{C}{x}$

$1-\left(\dfrac{y}{x}\right)^2 = \dfrac{C}{x}$

$\left[1-\dfrac{y^2}{x^2} = \dfrac{C}{x}\right]x^2$

$\boxed{x^2-y^2 = Cx}$

39) $y' = \dfrac{xy}{x^2-y^2}$

$\dfrac{dy}{dx} = \dfrac{xy}{x^2-y^2} \dfrac{\cdot \frac{1}{x^2}}{\cdot \frac{1}{x^2}} = \dfrac{\frac{y}{x}}{1-\left(\frac{y}{x}\right)^2}$

$\frac{y}{x} = v$

$vx = y$

$\frac{dy}{dx} = \frac{dv}{dx}x + v(1)$

$\dfrac{dv}{dx}x + v = \dfrac{v}{1-v^2}$

$\dfrac{dv}{dx}x = \dfrac{v}{1-v^2} - v$

$\dfrac{dv}{dx}x = \dfrac{v - v(1-v^2)}{1-v^2}$

$\dfrac{dv}{dx}x = \dfrac{v - v + v^3}{1-v^2}$

$\dfrac{dv}{dx}x = \dfrac{v^3}{1-v^2}$

$\dfrac{1-v^2}{v^3} dv = \dfrac{1}{x} dx$

$\int \dfrac{1-v^2}{v^3} dv = \int \dfrac{1}{x} dx$

$\int \dfrac{1}{v^3} - \dfrac{v^2}{v^3} dv = \int \dfrac{1}{x} dx$

$\int v^{-3} - \dfrac{1}{v} dv = \int \dfrac{1}{x} dx$

$\dfrac{v^{-2}}{-2} - \ln|v| = \ln|x| + c$

$\dfrac{-1}{2v^2} = \ln|x| + \ln|v| + \ln|c|$

$\dfrac{-1}{2v^2} = \ln|xvc|$

$-\dfrac{1}{2\left(\frac{y}{x}\right)^2} = \ln\left|x\left(\frac{y}{x}\right)c\right|$

$-\dfrac{1}{2\left(\frac{y^2}{x^2}\right)} = \ln|yc|$

$-\dfrac{x^2}{2y^2} = \ln|yc|$

$e^{-\frac{x^2}{2y^2}} = e^{\ln|yc|}$

$e^{-x^2/2y^2} = y \cdot C$

$y \cdot C = e^{-x^2/2y^2}$

$\boxed{y = Ce^{-x^2/2y^2}}$

40) $y' = \dfrac{2x+3y}{x}$ $\qquad \dfrac{dy}{dx} = \dfrac{2x+3y}{x} \dfrac{\cdot \frac{1}{x}}{\cdot \frac{1}{x}} = \dfrac{2+3\left(\frac{y}{x}\right)}{1}$

homogeneous degree: $n=1$

$v = \dfrac{y}{x}$

$\dfrac{dy}{dx} = \dfrac{dv}{dx}x + v$

$\dfrac{dv}{dx}x + v = 2+3v$

$\dfrac{dv}{dx}x = 2+2v$

$\dfrac{dv}{dx}x = 2(1+v)$

$\displaystyle\int \frac{dv}{1+v} = \int \frac{2}{x}\,dx$

$\ln|1+v| = 2\ln|x| + C$

$e^{\ln|1+v|} = e^{\ln|x|^2 + C}$

$|1+v| = x^2 \cdot C$

$|1+v| = Cx^2$

$\left[1 + \dfrac{y}{x} = Cx^2\right]x$

$x + y = Cx^3$

$\boxed{y = Cx^3 - x}$