BC Calculus **Ch. 6.3 Notes** **Homogeneous Differential Equations**

Differential equations needs to be separable in order for us to use available Integral rules to solve.

Not all differential equations are separable: Example is $\frac{dy}{dx} = x - 2y$

For differential equations that are not separable, we can use Slope Fields to graph potential solutions even though Integral Rules are insufficient to solve.

Some differential equations that are initially not separable can be made to be separable through the use of a change of variables. This is true for the differential equation of the form $y' = f(x, y)$, where f is **a homogeneous function**. The function is given by $\boldsymbol{f(x, y)}$ is homogeneous of **degree n** if:

$$\boldsymbol{f(tx, ty) = t^n f(x, y)} \quad \text{where n is a real number}$$

To determine if a function is homogeneous,

a) Substitute into the equation , $\boldsymbol{tx}$ *for each x, and* $\boldsymbol{ty}$ *for each y*
b) Simplify the equation and attempt to factor out t^n and rewrite as $\boldsymbol{t^n f(x, y)}$
c) If t^n can be factored out, then the function is homogeneous with a degree of n.

Determine if the function is homoegeneous. If so, determine the degree

Ex. 1: $f(x, y) = x^2 + y^2$

Ex. 2: $f(x, y) = x^2y - 4x^3 + 3xy^3 + 1$

Ex. 3: $f(x, y) = \frac{x^2+y^2}{\sqrt{x^2+y^2}}$

Homogeneous Differential Equation:

A homogeneous differential equation is an equation of the form $M(x, y)dx + n(x, y)dy = 0$ where M and N are homogeneous functions of the same degree

Steps for Solving Homogeneous Differential Equations:

1. Verify that differential equation $\frac{dy}{dx} = f(x, y)$ is homogeneous, and determine the degree n.
2. Multiply the numerator and denominator of differential equation by $\frac{1}{x^n}$
3. Substitute all $\frac{y}{x}$ with variable v $(v = \frac{y}{x})$
4. Through change of variable, solve for y, so that y = vx
5. Solve for $\frac{dy}{dx}$ using product rule, resulting in $\frac{dy}{dx} = \frac{dv}{dx}(x) + v(1)$

$$\frac{dy}{dx} = \frac{dv}{dx}x + v$$

6. Replace $\frac{dy}{dx}$ in the differential equation with the above expression $\frac{dv}{dx}x + v$
7. Solve the resulting differential equation using separation of variables (v,dv and x, dx)
8. Replace v back with original variables $\frac{y}{x}$ and solve for y.

Solve the homogeneous differential equation (Ch. 6.3 pg. 429 #35)

Ex. 4: $y' = \frac{x+y}{2x}$

Key

BC Calculus Ch. 6.3 Notes Homogeneous Differential Equations

Differential equations needs to be separable in order for us to use available Integral rules to solve.

Not all differential equations are separable: Example is $\frac{dy}{dx} = x - 2y$

For differential equations that are not separable, we can use Slope Fields to graph potential solutions even though Integral Rules are insufficient to solve.

Some differential equations that are initially not separable can be made to be separable through the use of a change of variables. This is true for the differential equation of the form $y' = f(x, y)$, where f is **a homogeneous function**. The function is given by $\boldsymbol{f(x, y)}$ is homogeneous of **degree n** if:

$$\boldsymbol{f(tx, ty) = t^n f(x, y)} \quad \text{where n is a real number}$$

To determine if a function is homogeneous,

a) Substitute into the equation $\boldsymbol{tx}$ *for each x, and* $\boldsymbol{ty}$ *for each y*
b) Simplify the equation and attempt to factor out t^n and rewrite as $\boldsymbol{t^n * f(x, y)}$
c) If t^n can be factored out, then the function is homogeneous with a degree of n.

Determine if the function is homoegeneous. If so, determine the degree:

Ex. 1: $f(x, y) = x^2 + y^2$

$$f(tx, ty) = (tx)^2 + (ty)^2$$
$$= t^2x^2 + t^2y^2$$
$$= t^2(x^2 + y^2) \checkmark$$

f(x,y) is a homogeneous function of degree 2.

Ex. 2: $f(x, y) = x^2y - 4x^3 + 3xy^3 + 1$

$$f(tx, ty) = (tx)^2(ty) - 4(tx)^3 + 3(tx)(ty)^3 + 1$$
$$= \underline{\underline{t^3}}x^2y - \underline{\underline{t^3}}(4x^3) + \underline{\underline{t^3}}(3xy^3) + 1$$
$$f(tx, ty) \neq t^n(x^2y - 4x^3 + 3xy^3 + 1)$$

f(x,y) is not a homogeneous function.

Ex. 3: $f(x, y) = \frac{x^2+y^2}{\sqrt{x^2+y^2}}$

$$f(tx, ty) = \frac{(tx)^2 + (ty)^2}{\sqrt{(tx)^2 + (ty)^2}}$$
$$= \frac{t^2(x^2+y^2)}{\sqrt{t^2(x^2+y^2)}}$$
$$= \frac{t^2}{t}\left[\frac{x^2+y^2}{\sqrt{x^2+y^2}}\right]$$
$$= t^1\left[\frac{x^2+y^2}{\sqrt{x^2+y^2}}\right]$$

f(x,y) is a homogeneous function of degree 1.

Steps for Solving Homogeneous Differential Equations:

1. Verify that differential equation $\frac{dy}{dx} = f(x, y)$ is homogeneous, and determine the degree n.
2. Multiply the numerator and denominator of differential equation by $\frac{1}{x^n}$
3. Substitute all $\frac{y}{x}$ with variable v $\left(v = \frac{y}{x}\right)$
4. Through change of variable, solve for y, so that y = vx
5. Solve for $\frac{dy}{dx}$ using product rule, resulting in $\frac{dy}{dx} = \frac{dv}{dx}(x) + v(1)$

$$\frac{dy}{dx} = \frac{dv}{dx}x + v$$

6. Replace $\frac{dy}{dx}$ in the differential equation with the above expression $\frac{dv}{dx}x + v$
7. Solve the resulting differential equation using separation of variables (v,dv and x, dx)
8. Replace v back with original variables $\frac{y}{x}$ and solve for y.

Solve the homogeneous differential equation (#35 p. 429)

$v = \frac{y}{x} \quad y = vx$

$\frac{dy}{dx} = \frac{dv}{dx}(x) + v(1)$

Ex. 4: $y' = \frac{x+y}{2x}$ degree : n=1

$$\frac{dy}{dx} = \frac{x+y}{2x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \frac{1 + \frac{y}{x}}{2} = \frac{1+v}{2}$$

$$x\frac{dv}{dx} + v = \frac{1+v}{2}$$

$$x\frac{dv}{dx} = \frac{1+v}{2} - v$$

$$x\frac{dv}{dx} = \frac{1+v-2v}{2}$$

$$x\frac{dv}{dx} = \frac{1-v}{2}$$

$$\frac{dv}{1-v} = \frac{1}{2} \cdot \frac{1}{x}dx$$

$$\int \frac{dv}{1-v} = \frac{1}{2}\int \frac{1}{x}dx$$

$u = 1-v$

$\frac{du}{dv} = -1$

$dv = -du$

$$\int \frac{-du}{u} = \frac{1}{2}\int \frac{1}{x}dx$$

$$-\ln|1-v| = \frac{1}{2}\ln|x| + C$$

$$\ln|1-v|^{-1} = \ln|x|^{1/2} + \ln|C|$$

$$e^{\ln|1-v|^{-1}} = e^{\ln|x^{1/2}C|}$$

$$(1-v)^{-1} = Cx^{1/2}$$

$$\left(1 - \frac{y}{x}\right)^{-1} = Cx^{1/2}$$

$$\left(\frac{x-y}{x}\right)^{-1} = Cx^{1/2}$$

$$\frac{x}{x-y} = Cx^{1/2}$$

$$\left(\frac{x}{x-y}\right)^2 = \left(Cx^{1/2}\right)^2$$

$$\frac{x^2}{(x-y)^2} = Cx$$

$$x^2 = Cx(x-y)^2$$

$$\boxed{x = C(x-y)^2}$$