64. $93 = 10\log_{10}\frac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-6.7 = \log_{10} I \Rightarrow I = 10^{-6.7}$

$80 = 10\log_{10}\frac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-8 = \log_{10} I \Rightarrow I = 10^{-8}$

Percentage decrease: $\left(\frac{10^{-6.7} - 10^{-8}}{10^{-6.7}}\right)(100) \approx 95\%$

65. Because $\frac{dy}{dt} = k(y - 80)$

$\int \frac{1}{y - 80}\,dy = \int k\,dt$

$\ln(y - 80) = kt + C.$

When $t = 0$, $y = 1500$. So, $C = \ln 1420$.

When $t = 1$, $y = 1120$. So,

$k(1) + \ln 1420 = \ln(1120 - 80)$

$k = \ln 1040 - \ln 1420 = \ln\frac{104}{142}.$

So, $y = 1420e^{[\ln(104/142)]t} + 80.$

When $t = 5$, $y \approx 379.2°\text{F}$.

66. $\frac{dy}{dt} = k(y - 20)$

$y = 20 + Ce^{kt}$ (See Example 6.)

$160 = 20 + Ce^{k(0)} \Rightarrow C = 140$

$60 = 20 + 140e^{k(5)}$

$\frac{2}{7} = e^{5k}$

$k = \frac{1}{5}\ln\left(\frac{2}{7}\right) \approx -0.25055$

$30 = 20 + 140e^{(1/5)\ln(2/7)t}$

$\frac{1}{14} = e^{\ln(2/7)^{t/5}} = \left(\frac{2}{7}\right)^{t/5}$

$\ln\frac{1}{14} = \frac{t}{5}\ln\frac{2}{7}$

$t = \frac{5\ln\frac{1}{14}}{\ln\frac{2}{7}} = \frac{5\ln 14}{\ln\frac{7}{2}} \approx 10.53$ minutes

It will take $10.53 - 5 = 5.53$ minutes longer.

67. False. If $y = Ce^{kt}$, $y' = Cke^{kt} \neq$ constant.

68. True

69. False. The prices are rising at a rate of 6.2% per year.

70. True

Section 6.3 Separation of Variables and the Logistic Equation

1. $\frac{dy}{dx} = \frac{x}{y}$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C_1$

$y^2 - x^2 = C$

2. $\frac{dy}{dx} = \frac{3x^2}{y^2}$

$\int y^2\,dy = \int 3x^2\,dx$

$\frac{y^3}{3} = x^3 + C_1$

$y^3 - 3x^3 = C$

3. $x^2 + 5y\frac{dy}{dx} = 0$

$5y\frac{dy}{dx} = -x^2$

$\int 5y\,dy = \int -x^2\,dx$

$\frac{5y^2}{2} = \frac{-x^3}{3} + C_1$

$15y^2 + 2x^3 = C$

4. $\frac{dy}{dx} = \frac{6 - x^2}{2y^3}$

$\int 2y^3\,dy = \int (6 - x^2)\,dx$

$\frac{y^4}{2} = 6x - \frac{x^3}{3} + C_1$

$3y^4 + 2x^3 - 36x = C$

5. $\frac{dr}{ds} = 0.75\,r$

$$\int \frac{dr}{r} = \int 0.75\,ds$$
$$\ln|r| = 0.75\,s + C_1$$
$$r = e^{0.75s + C_1}$$
$$r = Ce^{0.75s}$$

6. $\frac{dr}{ds} = 0.75\,s$

$$\int dr = \int 0.75\,s\,ds$$
$$r = 0.75\frac{s^2}{2} + C$$
$$r = 0.375\,s^2 + C$$

7. $(2 + x)y' = 3y$

$$\int \frac{dy}{y} = \int \frac{3}{2 + x}\,dx$$
$$\ln|y| = 3\ln|2 + x| + \ln C = \ln\left|C(2 + x)^3\right|$$
$$y = C(x + 2)^3$$

8. $xy' = y$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$
$$\ln y = \ln x + \ln C = \ln Cx$$
$$y = Cx$$

9. $yy' = 4\sin x$

$$y\frac{dy}{dx} = 4\sin x$$
$$\int y\,dy = \int 4\sin x\,dx$$
$$\frac{y^2}{2} = -4\cos x + C_1$$
$$y^2 = C - 8\cos x$$

10. $yy' = -8\cos(\pi x)$

$$y\frac{dy}{dx} = -8\cos(\pi x)$$
$$\int y\,dy = \int -8\cos(\pi x)\,dx$$
$$\frac{y^2}{2} = \frac{-8\sin(\pi x)}{\pi} + C$$
$$y^2 = \frac{-16}{\pi}\sin(\pi x) + C$$

11. $\sqrt{1 - 4x^2}\,y' = x$

$$dy = \frac{x}{\sqrt{1 - 4x^2}}\,dx$$
$$\int dy = \int \frac{x}{\sqrt{1 - 4x^2}}\,dx$$
$$= -\frac{1}{8}\int \left(1 - 4x^2\right)^{-1/2}(-8x\,dx)$$
$$y = -\frac{1}{4}\sqrt{1 - 4x^2} + C$$

12. $\sqrt{x^2 - 16}\,y' = 11x$

$$\frac{dy}{dx} = \frac{11x}{\sqrt{x^2 - 16}}$$
$$\int dy = \int \frac{11x}{\sqrt{x^2 - 16}}\,dx$$
$$y = 11\sqrt{x^2 - 16} + C$$

13. $y\ln x - xy' = 0$

$$\int \frac{dy}{y} = \int \frac{\ln x}{x}\,dx \quad \left(u = \ln x,\ du = \frac{dx}{x}\right)$$
$$\ln|y| = \frac{1}{2}(\ln x)^2 + C_1$$
$$y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$$

14. $12yy' - 7e^x = 0$

$$12y\frac{dy}{dx} = 7e^x$$
$$\int 12y\,dy = \int 7e^x\,dx$$
$$6y^2 = 7e^x + C$$

15. $yy' - 2e^x = 0$

$$y\frac{dy}{dx} = 2e^x$$
$$\int y\,dy = \int 2e^x\,dx$$
$$\frac{y^2}{2} = 2e^x + C$$

Initial condition $(0, 3)$: $\frac{9}{2} = 2 + C \Rightarrow C = \frac{5}{2}$

Particular solution: $\frac{y^2}{2} = 2e^x + \frac{5}{2}$

$$y^2 = 4e^x + 5$$

16. $\sqrt{x} + \sqrt{y}y' = 0$

$$\int y^{1/2}dy = -\int x^{1/2}\,dx$$
$$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + C_1$$
$$y^{3/2} + x^{3/2} = C$$

Initial condition $(1, 9)$:

$$(9)^{3/2} + (1)^{3/2} = 27 + 1 = 28 = C$$

Particular solution: $y^{3/2} + x^{3/2} = 28$

17. $y(x + 1) + y' = 0$

$$\int\frac{dy}{y} = -\int(x + 1)\,dx$$
$$\ln|y| = -\frac{(x + 1)^2}{2} + C_1$$
$$y = Ce^{-(x+1)^2/2}$$

Initial condition $(-2, 1)$: $1 = Ce^{-1/2}, C = e^{1/2}$

Particular solution: $y = e^{\left[1-(x+1)^2\right]/2} = e^{-\left(x^2+2x\right)/2}$

18. $2xy' - \ln x^2 = 0$

$$2x\frac{dy}{dx} = 2\ln x$$
$$\int dy = \int\frac{\ln x}{x}\,dx$$
$$y = \frac{(\ln x)^2}{2} + C$$

Initial condition $(1, 2)$: $2 = C$

Particular solution: $y = \frac{1}{2}(\ln x)^2 + 2$

19. $y(1 + x^2)y' = x(1 + y^2)$

$$\frac{y}{1 + y^2}\,dy = \frac{x}{1 + x^2}\,dx$$
$$\frac{1}{2}\ln(1 + y^2) = \frac{1}{2}\ln(1 + x^2) + C_1$$
$$\ln(1 + y^2) = \ln(1 + x^2) + \ln C = \ln\left[C(1 + x^2)\right]$$
$$1 + y^2 = C(1 + x^2)$$

Initial condition $(0, \sqrt{3})$: $1 + 3 = C \Rightarrow C = 4$

Particular solution: $1 + y^2 = 4(1 + x^2)$

$$y^2 = 3 + 4x^2$$

20. $y\sqrt{1 - x^2}\,\frac{dy}{dx} = x\sqrt{1 - y^2}$

$$\int(1 - y^2)^{-1/2}y\,dy = \int(1 - x^2)^{-1/2}x\,dx$$
$$-(1 - y^2)^{1/2} = -(1 - x^2)^{1/2} + C$$

Initial condition $(0, 1)$: $0 = -1 + C \Rightarrow C = 1$

Particular solution: $\sqrt{1 - y^2} = \sqrt{1 - x^2} - 1$

21. $\frac{du}{dv} = uv\sin v^2$

$$\int\frac{du}{u} = \int v\sin v^2\,dv$$
$$\ln|u| = -\frac{1}{2}\cos v^2 + C_1$$
$$u = Ce^{-\left(\cos v^2\right)/2}$$

Initial condition: $u(0) = 1$: $C = \frac{1}{e^{-1/2}} = e^{1/2}$

Particular solution: $u = e^{\left(1-\cos v^2\right)/2}$

22. $\frac{dr}{ds} = e^{r-2s}$

$$\int e^{-r}dr = \int e^{-2s}ds$$
$$-e^{-r} = -\frac{1}{2}e^{-2s} + C$$

Initial condition:

$$r(0) = 0: -1 = -\frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$$

Particular solution:

$$-e^{-r} = -\frac{1}{2}e^{-2s} - \frac{1}{2}$$
$$e^{-r} = \frac{1}{2}e^{-2s} + \frac{1}{2}$$
$$-r = \ln\left(\frac{1}{2}e^{-2s} + \frac{1}{2}\right) = \ln\left(\frac{1 + e^{-2s}}{2}\right)$$
$$r = \ln\left(\frac{2}{1 + e^{-2s}}\right)$$

23. $dP - kP\,dt = 0$

$$\int\frac{dP}{P} = k\int dt$$
$$\ln|P| = kt + C_1$$
$$P = Ce^{kt}$$

Initial condition: $P(0) = P_0, P_0 = Ce^0 = C$

Particular solution: $P = P_0e^{kt}$

24. $dT + k(T - 70)\,dt = 0$

$$\int \frac{dT}{T - 70} = -k\int dt$$
$$\ln(T - 70) = -kt + C_1$$
$$T - 70 = Ce^{-kt}$$

Initial condition:

$T(0) = 140$: $140 - 70 = 70 = Ce^0 = C$

Particular solution:

$T - 70 = 70e^{-kt}, T = 70(1 + e^{-kt})$

25. $y' = \frac{dy}{dx} = \frac{x}{4y}$

$$\int 4y\,dy = \int x\,dx$$
$$2y^2 = \frac{x^2}{2} + C$$

Initial condition $(0, 2)$: $2(2^2) = 0 + C \Rightarrow C = 8$

Particular solution: $2y^2 = \frac{x^2}{2} + 8$

$$4y^2 - x^2 = 16$$

26. $\frac{dy}{dx} = \frac{-9x}{16y}$

$$\int 16y\,dy = -\int 9x\,dx$$
$$8y^2 = \frac{-9}{2}x^2 + C$$

Initial condition $(1, 1)$: $8 = -\frac{9}{2} + C, C = \frac{25}{2}$

Particular solution: $8y^2 = \frac{-9}{2}x^2 + \frac{25}{2}$

$$16y^2 + 9x^2 = 25$$

27. $y' = \frac{dy}{dx} = \frac{y}{2x}$

$$\int \frac{2}{y}\,dy = \int \frac{1}{x}\,dx$$
$$2\ln|y| = \ln|x| + C_1 = \ln|x| + \ln C$$
$$y^2 = Cx$$

Initial condition $(9, 1)$: $1 = 9C \Rightarrow C = \frac{1}{9}$.

Particular solution: $y^2 = \frac{1}{9}x$

$$9y^2 - x = 0$$
$$y = \frac{1}{3}\sqrt{x}$$

28. $\frac{dy}{dx} = \frac{2y}{3x}$

$$\int \frac{3}{y}\,dy = \int \frac{2}{x}\,dx$$
$$\ln y^3 = \ln x^2 + \ln C$$
$$y^3 = Cx^2$$

Initial condition $(8, 2)$: $2^3 = C(8^2), C = \frac{1}{8}$

Particular solution: $8y^3 = x^2, y = \frac{1}{2}x^{2/3}$

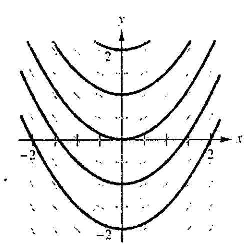
29. $m = \frac{dy}{dx} = \frac{0 - y}{(x + 2) - x} = -\frac{y}{2}$

$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dx$$
$$\ln|y| = -\frac{1}{2}x + C_1$$
$$y = Ce^{-x/2}$$

30. $m = \frac{dy}{dx} = \frac{y - 0}{x - 0} = \frac{y}{x}$

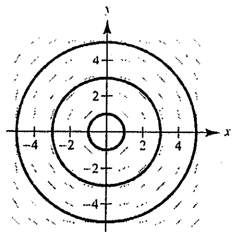
$$\int \frac{dy}{y} = \int \frac{dx}{x}$$
$$\ln y = \ln x + C_1 = \ln x + \ln C = \ln Cx$$
$$y = Cx$$

31. $\frac{dy}{dx} = x$



$$y = \int x\,dx = \frac{1}{2}x^2 + C$$

32. $\dfrac{dy}{dx} = -\dfrac{x}{y}$



$$y\,dy = -x\,dx$$
$$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$$
$$y^2 + x^2 = C$$

33. (a) $\dfrac{dy}{dx} = k(y - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $y = 4$; but not along $y = 0$. Matches (a).

34. (a) $\dfrac{dy}{dx} = k(x - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $x = 4$. Matches (b).

35. (a) $\dfrac{dy}{dx} = ky(y - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $y = 0$ and $y = 4$. Matches (c).

36. (a) $\dfrac{dy}{dx} = ky^2$

(b) The direction field satisfies $(dy/dx) = 0$ along $y = 0$, and grows more positive as y increases. Matches (d).

37. $\dfrac{dy}{dt} = ky, \quad y = Ce^{kt}$

Initial amount: $y(0) = y_0 = C$

Half-life: $\dfrac{y_0}{2} = y_0 e^{k(1599)}$

$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$$

$y = Ce^{[\ln(1/2)/1599]t}$

When $t = 50$, $y = 0.9786C$ or 97.86%.

38. $\dfrac{dy}{dt} = ky, \quad y = Ce^{kt}$

Initial conditions: $y(0) = 40$, $y(1) = 35$

$$40 = Ce^0 = C$$
$$35 = 40e^k$$
$$k = \ln\frac{7}{8}$$

Particular solution: $y = 40e^{t\ln(7/8)}$

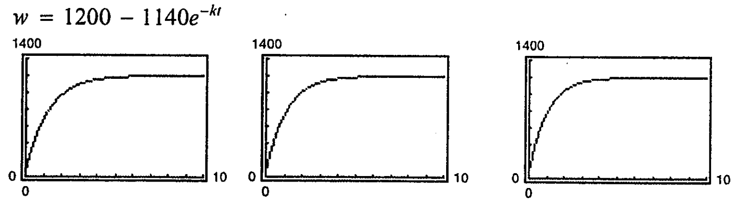
When 75% has been changed:

$$10 = 40e^{t\ln(7/8)}$$
$$\frac{1}{4} = e^{t\ln(7/8)}$$
$$t = \frac{\ln(1/4)}{\ln(7/8)} \approx 10.38 \text{ hours}$$

39. (a) $\dfrac{dw}{dt} = k(1200 - w)$

$$\int \frac{dw}{1200 - w} = \int k\,dt$$
$$\ln|1200 - w| = -kt + C_1$$
$$1200 - w = e^{-kt + C_1} = Ce^{-kt}$$
$$w = 1200 - Ce^{-kt}$$
$$w(0) = 60 = 1200 - C \Rightarrow C = 1200 - 60 = 1140$$
$$w = 1200 - 1140e^{-kt}$$

(b)



$k = 0.8$ $\quad$ $k = 0.9$ $\quad$ $k = 1$

(c) $k = 0.8$: $t = 1.31$ years

$k = 0.9$: $t = 1.16$ years

$k = 1.0$: $t = 1.05$ years

(d) Maximum weight: 1200 pounds

$\lim_{x\to\infty} w = 1200$

40. From Exercise 39

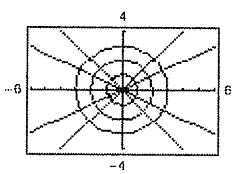
$$w = 1200 - Ce^{-kt}, k = 1$$
$$w = 1200 - Ce^{-t}$$
$$w(0) = w_0 = 1200 - C \Rightarrow C = 1200 - w_0$$
$$w = 1200 - (1200 - w_0)e^{-t}$$

41. Given family (circles): $x^2 + y^2 = C$

$$2x + 2yy' = 0$$
$$y' = -\frac{x}{y}$$

Orthogonal trajectory (lines): $y' = \frac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$
$$\ln|y| = \ln|x| + \ln K$$
$$y = Kx$$

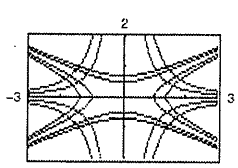


42. Given family (hyperbolas): $x^2 - 2y^2 = C$

$$2x - 4yy' = 0$$
$$y' = \frac{x}{2y}$$

Orthogonal trajectory: $y' = \frac{-2y}{x}$

$$\int \frac{dy}{y} = -\int \frac{2}{x}\,dx$$
$$\ln y = -2\ln x + \ln k$$
$$y = kx^{-2} = \frac{k}{x^2}$$

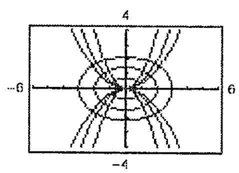


43. Given family (parabolas): $x^2 = Cy$

$$2x = Cy'$$
$$y' = \frac{2x}{C} = \frac{2x}{x^2/y} = \frac{2y}{x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{x}{2y}$

$$2\int y\,dy = -\int x\,dx$$
$$y^2 = -\frac{x^2}{2} + K_1$$
$$x^2 + 2y^2 = K$$

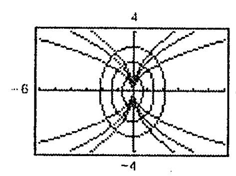


44. Given family (parabolas): $y^2 = 2Cx$

$$2yy' = 2C$$
$$y' = \frac{C}{y} = \frac{y^2}{2x}\left(\frac{1}{y}\right) = \frac{y}{2x}$$

Orthogonal trajectory (ellipse): $y' = -\frac{2x}{y}$

$$\int y\,dy = -\int 2x\,dx$$
$$\frac{y^2}{2} = -x^2 + K_1$$
$$2x^2 + y^2 = K$$

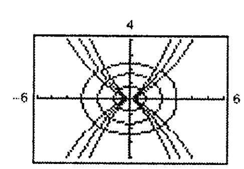


45. Given family: $y^2 = Cx^3$

$$2yy' = 3Cx^2$$
$$y' = \frac{3Cx^2}{2y} = \frac{3x^2}{2y}\left(\frac{y^2}{x^3}\right) = \frac{3y}{2x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{2x}{3y}$

$$3\int y\,dy = -2\int x\,dx$$
$$\frac{3y^2}{2} = -x^2 + K_1$$
$$3y^2 + 2x^2 = K$$



46. Given family (exponential functions): $y = Ce^x$

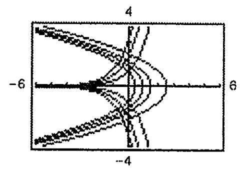
$$y' = Ce^x = y$$

Orthogonal trajectory (parabolas): $y' = -\frac{1}{y}$

$$\int y\,dy = -\int dx$$

$$\frac{y^2}{2} = -x + K_1$$

$$y^2 = -2x + K$$



47. $y = \dfrac{12}{1 + e^{-x}}$

Because $y(0) = 6$, it matches (c) or (d).

Because (d) approaches its horizontal asymptote slower than (c), it matches (d).

48. $y = \dfrac{12}{1 + 3e^{-x}}$

Because $y(0) = \frac{12}{4} = 3$, it matches (a).

49. $y = \dfrac{12}{1 + \frac{1}{2}e^{-x}}$

Because $y(0) = \dfrac{12}{\left(\frac{3}{2}\right)} = 8$, it matches (b).

50. $y = \dfrac{12}{1 + e^{-2x}}$

Because $y(0) = 6$, it matches (c) or (d).

Because y approaches $L = 12$ faster for (c), it matches (c).

51. $P(t) = \dfrac{2100}{1 + 29e^{-0.75t}}$

(a) $k = 0.75$

(b) $L = 2100$

(c) $P(0) = \dfrac{2100}{1 + 29} = 70$

(d)
$$1050 = \frac{2100}{1 + 29e^{-0.75t}}$$
$$1 + 29e^{-0.75t} = 2$$
$$e^{-0.75t} = \frac{1}{29}$$
$$-0.75t = \ln\left(\frac{1}{29}\right) = -\ln 29$$
$$t = \frac{\ln 29}{0.75} \approx 4.4897 \text{ yr}$$

(e) $\dfrac{dP}{dt} = 0.75P\left(1 - \dfrac{P}{2100}\right), \quad P(0) = 70$

52. $P(t) = \dfrac{5000}{1 + 39e^{-0.2t}}$

(a) $k = 0.2$

(b) $L = 5000$

(c) $P(0) = \dfrac{5000}{1 + 39} = 125$

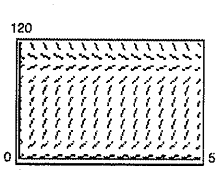
(d)
$$2500 = \frac{5000}{1 + 39e^{-0.2t}}$$
$$1 + 39e^{-0.2t} = 2$$
$$e^{-0.2t} = \frac{1}{39}$$
$$-0.2t = \ln\left(\frac{1}{39}\right) = -\ln 39$$
$$t = \frac{\ln 39}{0.2} \approx 18.3178$$

(e) $\dfrac{dP}{dt} = 0.2P\left(1 - \dfrac{P}{5000}\right), \quad P(0) = 125$

53. $\dfrac{dP}{dt} = 3P\left(1 - \dfrac{P}{100}\right)$ (a) $k = 3$

(b) $L = 100$

(c)



(d) $\frac{d^2P}{dt^2} = 3P'\left(1 - \frac{P}{100}\right) + 3P\left(\frac{-P'}{100}\right)$

$= 3\left[3P\left(1 - \frac{P}{100}\right)\right]\left(1 - \frac{P}{100}\right) - \frac{3P}{100}\left[3P\left(1 - \frac{P}{100}\right)\right] = 9P\left(1 - \frac{P}{100}\right)\left(1 - \frac{P}{100} - \frac{P}{100}\right) = 9P\left(1 - \frac{P}{100}\right)\left(1 - \frac{2P}{100}\right)$

$\frac{d^2P}{dt^2} = 0$ for $P = 50$, and by the first Derivative Test, this is a maximum. $\left(\text{Note: } P = 50 = \frac{L}{2} = \frac{100}{2}\right)$

54. $\frac{dP}{dt} = 0.1P - 0.0004P^2$

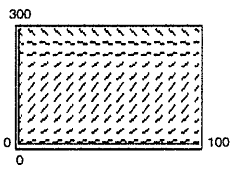
$= 0.1P(1 - 0.004P)$

$= 0.1P\left(1 - \frac{P}{250}\right)$

(a) $k = 0.1 = \frac{1}{10}$

(b) $L = 250$

(c)



(d) $P = \frac{250}{2} = 125$. (Same argument as in Exercise 77)

55. $\frac{dy}{dt} = y\left(1 - \frac{y}{36}\right), \quad y(0) = 4$

$k = 1, L = 36$

$y = \frac{L}{1 + be^{-kt}} = \frac{36}{1 + be^{-t}}$

$(0, 4)$: $4 = \frac{36}{1 + b} \Rightarrow b = 8$

Solution: $y = \frac{36}{1 + 8e^{-t}}$

56. $\frac{dy}{dt} = 2.8y\left(1 - \frac{y}{10}\right), \quad y(0) = 7$

$k = 2.8, L = 10$

$y = \frac{L}{1 + be^{-kt}} = \frac{10}{1 + be^{-2.8t}}$

$(0, 7)$: $7 = \frac{10}{1 + b} \Rightarrow 1 + b = \frac{10}{7} \Rightarrow b = \frac{3}{7}$

Solution: $y = \frac{10}{1 + \left(\frac{3}{7}\right)e^{-2.8t}}$

57. $\frac{dy}{dt} = \frac{4y}{5} - \frac{y^2}{150} = \frac{4}{5}y\left(1 - \frac{y}{120}\right), \quad y(0) = 8$

$k = \frac{4}{5} = 0.8, L = 120$

$y = \frac{L}{1 + be^{-kt}} = \frac{120}{1 + be^{-0.8t}}$

$(0, 8)$: $8 = \frac{120}{1 + b} \Rightarrow b = 14$

Solution: $y = \frac{120}{1 + 14e^{-0.8t}}$

58. $\frac{dy}{dt} = \frac{3y}{20} - \frac{y^2}{1600} = \frac{3}{20}y\left(1 - \frac{y}{240}\right), \quad y(0) = 15$

$k = \frac{3}{20}, L = 240$

$y = \frac{L}{1 + be^{-kt}} = \frac{240}{1 + be^{(-3/20)t}}$

$(0, 15)$: $15 = \frac{240}{1 + b} \Rightarrow b = 15$

Solution: $y = \frac{240}{1 + 15e^{(-3/20)t}}$

59. (a) $P = \dfrac{L}{1 + be^{-kt}}, L = 200, P(0) = 25$

$$25 = \frac{200}{1 + b} \Rightarrow b = 7$$

$$39 = \frac{200}{1 + 7e^{-k(2)}}$$

$$1 + 7e^{-2k} = \frac{200}{39}$$

$$e^{-2k} = \frac{23}{39}$$

$$k = -\frac{1}{2}\ln\left(\frac{23}{39}\right) = \frac{1}{2}\ln\left(\frac{39}{23}\right) \approx 0.2640$$

$$P = \frac{200}{1 + 7e^{-0.2640t}}$$

(b) For $t = 5$, $P \approx 70$ panthers.

(c)
$$100 = \frac{200}{1 + 7e^{-0.264t}}$$

$$1 + 7e^{-0.264t} = 2$$

$$-0.264t = \ln\left(\frac{1}{7}\right)$$

$$t \approx 7.37 \text{ years}$$

(d)
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

$$= 0.264P\left(1 - \frac{P}{200}\right),\ P(0) = 25$$

Using Euler's Method, $P \approx 65.6$ when $t = 5$.

(e) P is increasing most rapidly where $P = 200/2 = 100$, corresponds to $t \approx 7.37$ years.

60. (a)
$$y = \frac{L}{1 + be^{-kt}},\ L = 20,\ y(0) = 1,\ y(2) = 4$$

$$1 = \frac{20}{1 + b} \Rightarrow b = 19$$

$$4 = \frac{20}{1 + 19e^{-2k}}$$

$$1 + 19e^{-2k} = 5$$

$$19e^{-2k} = 4$$

$$k = -\frac{1}{2}\ln\left(\frac{4}{19}\right) = \frac{1}{2}\ln\left(\frac{19}{4}\right) \approx 0.7791$$

$$y = \frac{20}{1 + 19e^{-0.7791t}}$$

(b) For $t = 5$, $y \approx 14.43$ grams

(c)

$$18 = \frac{20}{1 + 19e^{-0.7791t}}$$

$$1 + 19e^{-0.7791t} = \frac{20}{18} = \frac{10}{9}$$

$$19e^{-0.7791t} = \frac{1}{9}$$

$$e^{-0.7791t} = \frac{1}{171}$$

$$t = \frac{-1}{0.7791}\ln\left(\frac{1}{171}\right) \approx 6.60 \text{ hours}$$

(d) $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) = \frac{1}{2}\ln\left(\frac{19}{4}\right)y\left(1 - \frac{y}{20}\right)$

t	0	1	2	3	4	5
Exact	1	2.06	4.00	7.05	10.86	14.43
Euler	1	1.74	2.98	4.95	7.86	11.57

(e) The weight is increasing most rapidly when $y = L/2 = 20/2 = 10$, corresponding to $t \approx 3.78$ hours.

61. A differential equation can be solved by separation of variables if it can be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0.$$

To solve a separable equation, rewrite as,

$$M(x)\,dx = -N(y)\,dy$$

and integrate both sides.

62. Two families of curves are mutually orthogonal if each curve in the first family intersects each curve in the second family at right angles.

63. $y = \frac{1}{1 + be^{-kt}}$

$$y' = \frac{-1}{\left(1 + be^{-kt}\right)^2}\left(-bke^{-kt}\right)$$

$$= \frac{k}{\left(1 + be^{-kt}\right)} \cdot \frac{be^{-kt}}{\left(1 + be^{-kt}\right)}$$

$$= \frac{k}{\left(1 + be^{-kt}\right)} \cdot \frac{1 + be^{-kt} - 1}{\left(1 + be^{-kt}\right)}$$

$$= \frac{k}{\left(1 + be^{-kt}\right)} \cdot \left(1 - \frac{1}{1 + be^{-kt}}\right) = ky(1 - y)$$

64. $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right),\ y(0) < L$

$$\frac{d^2y}{dt^2} = ky'\left(1 - \frac{y}{L}\right) + ky\left(-\frac{y'}{L}\right)$$

$$= k^2y\left(1 - \frac{y}{L}\right)^2 + ky\left[\frac{-ky\left(1 - \frac{y}{L}\right)}{L}\right]$$

$$= k^2\left(1 - \frac{y}{L}\right)y\left[\left(1 - \frac{y}{L}\right) - \frac{y}{L}\right]$$

$$= k^2\left(1 - \frac{y}{L}\right)y\left(1 - \frac{2y}{L}\right)$$

So, $\frac{d^2y}{dt^2} = 0$ when $1 - \frac{2y}{L} = 0 \Rightarrow y = \frac{L}{2}$.

By the First Derivative Test, this is a maximum.

65. (a)

$$\frac{dv}{dt} = k(W - v)$$

$$\int \frac{dv}{W - v} = \int k\,dt$$

$$-\ln|W - v| = kt + C_1$$

$$v = W - Ce^{-kt}$$

Initial conditions:

$W = 20, v = 0$ when $t = 0$ and $v = 10$ when $t = 0.5$ so, $C = 20, k = \ln 4$.

Particular solution:

$$v = 20\left(1 - e^{-(\ln 4)t}\right) = 20\left(1 - \left(\frac{1}{4}\right)^t\right)$$

or

$$v = 20\left(1 - e^{-1.386t}\right)$$

(b) $s = \int 20\left(1 - e^{-1.386t}\right)dt \approx 20\left(t + 0.7215e^{-1.386t}\right) + C$

Because $s(0) = 0, C \approx -14.43$ and you have

$$s \approx 20t + 14.43\left(e^{-1.386t} - 1\right).$$

66. Answers will vary. *Sample answer*: There might be limits on available food or space.

67. $f(x, y) = x^3 - 4xy^2 + y^3$

$$f(tx, ty) = t^3x^3 - 4txt^2y^2 + t^3y^3$$
$$= t^3\left(x^3 - 4xy^2 + y^3\right)$$

Homogeneous of degree 3

68. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

$$f(tx, ty) = t^3x^3 + 3t^4x^2y^2 - 2t^2y^2$$

Not homogeneous

69. $f(x, y) = \dfrac{x^2y^2}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{t^4x^2y^2}{\sqrt{t^2x^2 + t^2y^2}} = t^3\frac{x^2y^2}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 3

70. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{tx\,ty}{\sqrt{t^2x^2 + t^2y^2}}$$
$$= \frac{t^2xy}{t\sqrt{x^2 + y^2}} = t\frac{xy}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 1

71. $f(x, y) = 2\ln xy$

$$f(tx, ty) = 2\ln[txty]$$
$$= 2\ln\left[t^2xy\right] = 2\left(\ln t^2 + \ln xy\right)$$

Not homogeneous

72. $f(x, y) = \tan(x + y)$

$$f(tx, ty) = \tan(tx + ty) = \tan[t(x + y)]$$

Not homogeneous

73. $f(x, y) = 2\ln\dfrac{x}{y}$

$$f(tx, ty) = 2\ln\frac{tx}{ty} = 2\ln\frac{x}{y}$$

Homogeneous of degree 0

74. $f(x, y) = \tan\dfrac{y}{x}$

$$f(tx, ty) = \tan\frac{ty}{tx} = \tan\frac{y}{x}$$

Homogeneous of degree 0

75. $(x + y)dx - 2x\,dy = 0, y = ux, dy = x\,du + u\,dx$

$$(x + ux)dx - 2x(x\,du + u\,dx) = 0$$
$$(1 + u)dx - 2x\,du - 2u\,dx = 0$$
$$(1 - u)dx = 2x\,du$$
$$\frac{1}{x}dx = \frac{2}{1 - u}du$$
$$\int \frac{1}{x}dx = 2\int \frac{1}{1 - u}du$$
$$\ln|x| + \ln C = -2\ln|1 - u|$$
$$\ln|Cx| = \ln|1 - u|^{-2}$$
$$|Cx| = \frac{1}{(1 - u)^2} = \frac{1}{[1 - (y/x)]^2}$$
$$|Cx| = \frac{x^2}{(x - y)^2}$$
$$|x| = C(x - y)^2$$

76. $(x^3 + y^3)dx - xy^2\,dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$\left[x^3 + (ux)^3\right]dx - x(ux)^2(x\,du + u\,dx) = 0$$
$$(1 + u^3)dx - u^2(x\,du + u\,dx) = 0$$
$$dx = xu^2\,du$$
$$\int\frac{dx}{x} = \int u^2\,du$$
$$\ln|x| + C_1 = \frac{u^3}{3} = \frac{1}{3}\left(\frac{y}{x}\right)^3$$
$$\left(\frac{y}{x}\right)^3 = 3\ln|x| + C$$
$$y^3 = 3x^3\ln|x| + Cx^3$$

77. $(x - y)dx - (x + y)dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$(x - ux)dx - (x + ux)(x\,du + u\,dx) = 0$$
$$(1 - u)dx - (1 + u)(x\,du + u\,dx) = 0$$
$$(1 - 2u - u^2)dx = x(1 + u)du$$
$$-\frac{dx}{x} = \frac{1 + u}{u^2 + 2u - 1}du$$
$$-\int\frac{dx}{x} = \int\frac{u + 1}{u^2 + 2u - 1}du$$
$$-\ln|x| + \ln C = \frac{1}{2}\ln\left|u^2 + 2u - 1\right|$$
$$\ln\left|\frac{C}{x}\right| = \ln\left|u^2 + 2u - 1\right|^{1/2}$$
$$\frac{C^2}{x^2} = \left|u^2 + 2u - 1\right|$$
$$\frac{C}{x^2} = \left|\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right) - 1\right|$$
$$C = \left|y^2 + 2yx - x^2\right|$$

78. $(x^2 + y^2)dx - 2x\,dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$\left(x^2 + (ux)^2\right)dx - 2x(ux)(x\,du + u\,dx) = 0$$
$$(1 + u^2)dx - 2u(x\,du + u\,dx) = 0$$
$$(1 - u^2)dx = 2ux\,du$$
$$-\frac{dx}{x} = \frac{-2u}{1 - u^2}du$$
$$-\int\frac{dx}{x} = \int\frac{-2u\,du}{1 - u^2}$$
$$-\ln|x| + \ln C = \ln\left|1 - u^2\right| = \ln\left[u^2 - 1\right] = \ln\left|u^2 - 1\right|$$
$$\ln\left|\frac{C}{x}\right| = \ln\left|u^2 - 1\right|$$
$$\frac{C}{x} = u^2 - 1 = \left(\frac{y}{x}\right)^2 - 1$$
$$Cx = y^2 - x^2$$

79. $xydx + (y^2 - x^2)dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$x(ux)\,dx + \left[(ux)^2 - x^2\right](x\,du + u\,dx) = 0$$
$$u\,dx + (u^2 - 1)(x\,du + u\,dx) = 0$$
$$u^3\,dx = -(u^2 - 1)x\,du$$
$$\frac{dx}{x} = \frac{1 - u^2}{u^3}\,du$$
$$\int\frac{dx}{x} = \int\left(u^{-3} - \frac{1}{u}\right)du$$
$$\ln|x| + \ln|C_1| = -\frac{1}{2u^2} - \ln|u|$$
$$\ln|C_1 xu| = -\frac{1}{2u^2}$$
$$\ln|C_1 y| = -\frac{1}{2(y/x)^2} = -\frac{x^2}{2y^2}$$
$$y = Ce^{-x^2/(2y^2)}$$

80. $(2x + 3y)dx - x\,dy = 0,\ y = ux,\ dy = x\,du + u\,dx$

$$(2x + 3ux)dx - x(x\,du + u\,dx) = 0$$
$$(2 + 3u)dx - x\,du - u\,dx = 0$$
$$(2 + 2u)dx = x\,du$$
$$\frac{2dx}{x} = \frac{du}{1 + u}$$
$$2\int\frac{1}{x}dx = \int\frac{1}{u + 1}\,du$$
$$2\ln|x| + \ln C = \ln|u + 1|$$
$$\ln x^2 C = \ln|u + 1|$$
$$1 + u = x^2 C$$
$$1 + \frac{y}{x} = x^2 C$$
$$\frac{y}{x} = Cx^2 - 1$$
$$y = Cx^3 - x$$

81. False. $\dfrac{dy}{dx} = \dfrac{x}{y}$ is separable, but $y = 0$ is not a solution.

82. True

$$\frac{dy}{dx} = (x - 2)(y + 1)$$

83. True

$$x^2 + y^2 = 2Cy \qquad x^2 + y^2 = 2Kx$$

$$\frac{dy}{dx} = \frac{x}{C - y} \qquad \frac{dy}{dx} = \frac{K - x}{y}$$

$$\frac{x}{C - y} \cdot \frac{K - x}{y} = \frac{Kx - x^2}{Cy - y^2} = \frac{2Kx - 2x^2}{2Cy - 2y^2} = \frac{x^2 + y^2 - 2x^2}{x^2 + y^2 - 2y^2} = \frac{y^2 - x^2}{x^2 - y^2} = -1$$

84. $fg' + gf' = f'g'$ Product Rule

$$(f - f')g' + gf' = 0$$

$$g' + \frac{f'}{f - f'}g = 0$$

Need $f - f' = e^{x^2} - 2xe^{x^2} = (1 - 2x)e^{x^2} \neq 0$, so avoid $x = \frac{1}{2}$.

$$\frac{g'}{g} = \frac{f'}{f' - f} = \frac{2xe^{x^2}}{(2x - 1)e^{x^2}} = 1 + \frac{1}{2x - 1}$$

$$\ln|g(x)| = x + \frac{1}{2}\ln|2x - 1| + C_1$$

$$g(x) = Ce^x|2x - 1|^{1/2}$$

So there exists g and interval (a, b), as long as $\frac{1}{2} \notin (a, b)$.

Section 6.4 First-Order Linear Differential Equations

1. $x^3y' + xy = e^x + 1$

$$y' + \frac{1}{x^2}y = \frac{1}{x^3}(e^x + 1)$$

Linear

2. $2xy - y' \ln x = y$

$$(\ln x)y' + (1 - 2x)y = 0$$

$$y' + \frac{(1 - 2x)}{\ln x}y = 0$$

Linear

3. $y' - y \sin x = xy^2$

Not linear, because of the xy^2-term.

4. $\frac{2 - y'}{y} = 5x$

$$2 - y' = 5xy$$

$$y' + 5xy = 2$$

Linear

5. $\frac{dy}{dx} + \left(\frac{1}{x}\right)y = 6x + 2$

Integrating factor: $e^{\int (1/x)\,dx} = e^{\ln x} = x$

$$xy = \int x(6x + 2)\,dx = 2x^3 + x^2 + C$$

$$y = 2x^2 + x + \frac{C}{x}$$

6. $\frac{dy}{dx} + \frac{2}{x}y = 3x - 5$

Integrating factor: $e^{\int 2/x\,dx} = e^{\ln x^2} = x^2$

$$x^2y = \int x^2(3x - 5)\,dx = \frac{3}{4}x^4 + \frac{5x^3}{3} + C$$

$$y = \frac{3}{4}x^2 + \frac{5}{3}x + \frac{C}{x^2}$$

7. $y' - y = 16$

Integrating factor: $e^{\int -1\,dx} = e^{-x}$

$$e^{-x}y' - e^{-x}y = 16e^{-x}$$

$$ye^{-x} = \int 16e^{-x}\,dx = -16e^{-x} + C$$

$$y = -16 + Ce^x$$

8. $y' + 2xy = 10x$

Integrating factor: $e^{\int 2x\,dx} = e^{x^2}$

$$ye^{x^2} = \int 10xe^{x^2}\,dx = 5e^{x^2} + C$$

$$y = 5 + Ce^{-x^2}$$