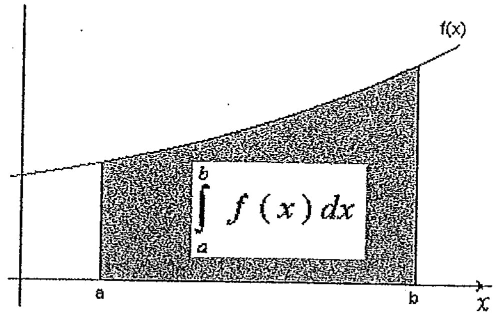
Calculus **Chapters** 6.3a **1st Fundamental Theorem of Calculus**

Key

$* \int_a^b f'(x)dx = f(b) - f(a)$

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is the antiderivative of f.



Recall:

*The general derivative is a **slope-finding function** or formula : (ex. $f'(x) = 2x + 1$)

*The specific derivative is the **actual slope** at a point (ex: $f'(3) = 7$)

Likewise…

The indefinite integral is an **Area-Finding Function** or formula (Ex: $\int 2x\,dx = x^2 + C$)

The definite integral is the **Actual Area** of the region for an interval (Ex: $\int_1^3 2x\,dx = 8$)

*If a function is **continuous** on a closed interval, then the function is able to be integrated on that interval

$$\int_a^b f'(x)dx = f(b) - f(a)$$

Class Examples:

1. Evaluate $\int_1^4 (3x^2 + 4x - 1)dx$

$$\left.\frac{3x^3}{3} + \frac{4x^2}{2} - x\right]_1^4$$

$$\left.x^3 + 2x^2 - x\right]_1^4$$

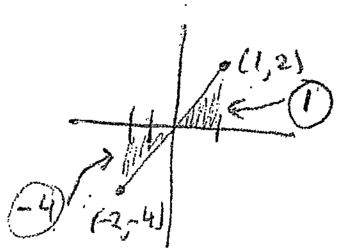
$$4^3 + 2(4)^2 - 4 - (1^3 + 2 - 1)$$

$$92 - 2 = \boxed{90}$$

For definite integrals:

**NOTE: we don't need to worry about the constant of integration " +C ". It will always wash out.

2. Evaluate $\int_{-2}^1 2x\,dx = \frac{2x^2}{2} = \left. x^2\right]_{-2}^{1} = 1^2 - (-2)^2 = \boxed{-3}$



portions of graph below x-axis will result in negative value.

Integral Properties:

1) $\int_a^a f(x)dx = 0$

2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ (given that c is between a and b)

Example 3: If $\int_0^3 f(x)dx = 4$ and $\int_3^6 f(x)dx = -1$, find the below:

a) $\int_0^6 f(x)dx = \int_0^3 f(x)dx + \int_3^6 f(x)dx = 4 + (-1) = \boxed{3}$

b) $\int_6^3 f(x)dx = -\int_3^6 f(x)dx = -(-1) = \boxed{1}$

c) $\int_3^3 f(x)dx = \boxed{0}$

d) $\int_3^6 (-5f(x)+3)dx = -5\int_3^6 f(x)dx + \int_3^6 3dx$

$3x\Big]_3^6 = 18 - 9 = 9$

$= -5(-1) + 9 = \boxed{14}$

Ex. 4: If $\int_3^8 f'(x)dx = 10$ and $f(8) = 6$, find $f(3)$.

*Reminder that the FFTC can be used as an equation solving tool to find the value of an antiderivative at a specific point. Recall that $\int_a^b f'(x)dx = f(b) - f(a)$

$\int_3^8 f'(x)dx = f(8) - f(3)$

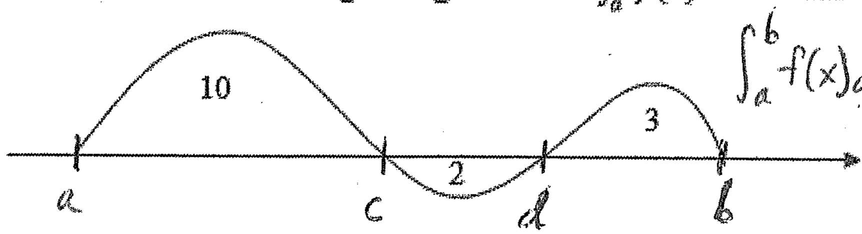
$10 = 6 - f(3)$

$10 - 6 = -f(3)$

$4 = -f(3)$

$\boxed{f(3) = -4}$

Ex. 5: The area for each region is given. Find $\int_a^b f(x)dx$



$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^b f(x)dx$

$= 10 + (-2) + 3$

$= \boxed{11}$

6.3b Notes – 2nd Fundamental Theorem of Calculus and Average Value Theorem

Key

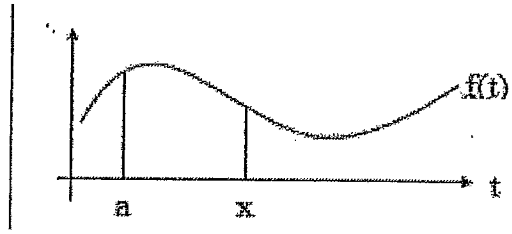
Definite Integral as a Function

To recap, we've covered:

1) Indefinite General Integrals (Area-finding functions)
2) Definite Integrals (Finds Area between 2 x-values)

There is also now a function that is the integral itself. Instead of going from a constant to another constant, we are going from a constant to a moving value of x.

Consider: $f(x)=\int_a^x f(t)dt$



2nd Fundamental Theorem of Calculus **Very Important**

Applies the concept that derivative and integrals are inverse operations of each other.

1) $\frac{d}{dx}\left[\int_a^{p(x)} f(t)dt\right]=f(p(x))\cdot p'(x)$ (a is a constant)

2) $\frac{d}{dx}\left[\int_{q(x)}^{p(x)} f(t)dt\right]=f(p(x))\cdot p'(x)-f(q(x))\cdot q'(x)$

Example 1:

$\frac{d}{dx}\left[\int_1^{x^2}(2t-3)dt\right] \rightarrow \frac{2t^2}{2}-3t \rightarrow t^2-3t\Big]_1^{x^2} \rightarrow (x^2)^2-3(x^2)-(1-3)$

$\frac{d}{dx}\left[x^4-3x^2+2\right] \rightarrow 4x^3-6x+0 \rightarrow \boxed{4x^3-6x}$

SFTC

$(2(x^2)-3)\cdot 2x \rightarrow \boxed{4x^3-6x}$

1) $\frac{d}{dx}\left[\int_a^{p(x)} f(t)dt\right] = f(p(x))\cdot p'(x)$ (a is a constant)

2) $\frac{d}{dx}\left[\int_{q(x)}^{p(x)} f(t)dt\right] = f(p(x))\cdot p'(x) - f(q(x))\cdot q'(x)$

Example 2:

a) $\frac{d}{dx}\left[\int_{-3}^{x} \sqrt{t^2+4}\,dt\right] = \sqrt{x^2+4}\cdot(1)$

$\boxed{\sqrt{x^2+4}}$

b) $\frac{d}{dx}\left[\int_{3}^{x^2} \sqrt{t-1}\,dt\right] = \sqrt{x^2-1}\cdot 2x$

$\boxed{2x\sqrt{x^2-1}}$

c) $\frac{d}{dx}\left[\int_{10}^{x^2} \sqrt{t-1}\,dt\right] = \sqrt{x^2-1}\cdot 2x$

$\boxed{2x\sqrt{x^2-1}}$

d) $\frac{d}{dx}\left[\int_{3x}^{0} \frac{1}{t+2}dt\right] = \frac{d}{dx}\int_0^{3x} \frac{-1}{t+2}dt$

$\frac{-1}{3x+2}\cdot 3 \rightarrow \boxed{\frac{-3}{3x+2}}$

e) $\frac{d}{dx}\left[\int_{x}^{x^2} (2t+3)dt\right] =$

$[2(x^2)+3]\cdot 2x - (2x+3)(1)$

$4x^3+6x-2x-3$

$\boxed{4x^3+4x-3}$

OR... $\frac{2t^2}{2}+3t \rightarrow t^2+3t\Big]_x^{x^2} \rightarrow x^4+3x^2-x^2-3x$

$\frac{d}{dx}(x^4+2x^2-3x)$

$= \boxed{4x^3+4x-3}$

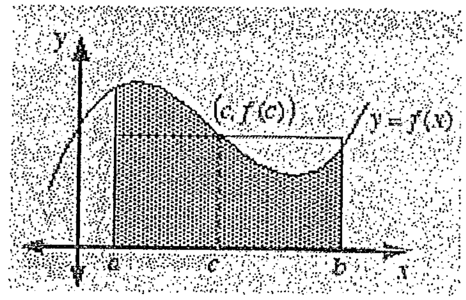
$\boxed{\text{Ex. 2}}$ Find $\frac{d}{dx}\left[\int_{2x^3}^{5} \frac{2t}{5-t^2}dt\right]$

$*$ $\frac{d}{dx}\int_a^{p(x)} f(t)dt = f(p(x))\,p'(x)$

$\frac{d}{dx}\int_5^{2x^3} \frac{-2t}{5-t^2}dt = \frac{-2(2x^3)}{5-(2x^3)^2}\cdot 6x^2 \rightarrow \boxed{\frac{-24x^5}{5-4x^6}}$

6.3b Average Value Theorem

If function f is integrable on the closed interval [a, b], then the average value of f on the interval is



$$\text{height} \rightarrow f(c) = \frac{1}{\underset{\text{width}}{b-a}} \boxed{\int_a^b f(x)dx} \text{ Area}$$

$$\text{height} = \frac{\text{Area}}{\text{width}} \rightarrow \frac{\int_a^b f(x)\,dx}{b-a}$$

$$\boxed{f(c) = \frac{1}{b-a}\int_a^b f(x)dx}$$

Rectangle Area = width × height

$$\frac{\text{Area}}{\text{width}} = \text{height}$$

*There exists a rectangle such that the area of the rectangle is the same as the area under the curve (shaded region). f (c) is the height of the rectangle

Example 1: a) Find the average value of $f(x) = x^2 + 1$ on [2, 5]. b) find the c value

$$f(c) = \frac{1}{5-2}\int_2^5 x^2+1\,dx \rightarrow \left.\frac{x^3}{3}+x\right]_2^5 \rightarrow \frac{5^3}{3}+5-\left(\frac{2^3}{3}+2\right) \rightarrow \frac{125}{3}+5-\frac{8}{3}-2$$

$$\frac{1}{3}\left[\frac{117}{3}+3\right] \rightarrow \frac{1}{3}\left(\frac{117}{3}+\frac{9}{3}\right) \rightarrow \frac{1}{3}\left(\frac{126}{3}\right) \rightarrow \frac{126}{9} \rightarrow 14 \quad \boxed{f(c)=14}$$

b) $14 = x^2 + 1$

$13 = x^2$

$x = \pm\sqrt{13}$

$\boxed{c = \sqrt{13}}$

Practice Problem:

Avg. value theorem: $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$

#47) $f(x) = 4 - x^2$ $[-2, 2]$ a) Find avg. value

b) find c-value

$$f(c) = \frac{1}{2-(-2)}\int_{-2}^{2} 4-x^2\,dx \rightarrow \left.4x-\frac{x^3}{3}\right]_{-2}^{2} \rightarrow 8-\frac{8}{3}-\left(-8+\frac{8}{3}\right) \rightarrow 16-\frac{16}{3}$$

$$f(c) = \frac{1}{4}\left[16-\frac{16}{3}\right]$$

$$f(c) = \frac{1}{4}\left[\frac{48-16}{3}\right] = \frac{1}{4}\cdot\frac{32}{3} = \frac{8}{3}$$

$f(c) = 8/3$

b) $\frac{8}{3} = 4 - x^2$

$x^2 = -\frac{8}{3} + \frac{12}{3}$

$x^2 = \frac{4}{3}$

$x = \pm\sqrt{\frac{4}{3}}$

$$\boxed{c = \frac{2}{\sqrt{3}},\quad c = -\frac{2}{\sqrt{3}}}$$

Key

Practice Problems: Average Value Theorem, First Theorem of Calculus, 2^{nd} Theorem of Calculus

For each problem, find the values of c that satisfy the Mean Value Theorem for Integrals.

1) $f(x) = 2x^2 - 4x - 4;\ [0, 3]$

$* f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

$f(c) = \frac{1}{3-0}\int_0^3 2x^2 - 4x - 4\,dx$

$\frac{2x^3}{3} - \frac{4x^2}{2} - 4x \rightarrow \left.\frac{2}{3}x^3 - 2x^2 - 4x\right]_0^3 = \frac{2}{3}(3)^3 - 2(3)^2 - 4(3) - 0$

$\frac{1}{3}\left[18 - 18 - 12\right] \rightarrow \frac{-12}{3} = -4$

$f(c) = -4$

$-4 = 2x^2 - 4x - 4$

$0 = 2x^2 - 4x + 0$

$0 = 2x(x-2)$

$\boxed{x=0,\ x=2}$

$\boxed{c=0,\ c=2}$

For each problem, find the average value of the function over the given interval.

2) $f(x) = -x^2 + 2x + 1;\ [1, 4]$

$f(c) = \frac{1}{4-1}\int_1^4 -x^2 + 2x + 1\,dx$

$\left.-\frac{x^3}{3} + \frac{2x^2}{2} + x\right]_1^4 = -\frac{4^3}{3} + 4^2 + 4 - \left(-\frac{1}{3} + 1 + 1\right)$

$\frac{1}{3}\left[-\frac{64}{3} + 20 + \frac{1}{3} - 2\right] = \frac{1}{3}(-21 + 18) = \frac{1}{3}(-3) = \boxed{-1}$

1. $\frac{d}{dx}\int_2^x \ln t\,dt =$

$\boxed{\text{(A)}}$ $\ln x$ (B) $\ln 2$ (C) $\frac{1}{x}$

(D) $\frac{1}{2}$ (E) $\ln x - \ln 2$

$\ln(x)\cdot(1) = \boxed{\ln x}$

2. If $g(x) = \int_\pi^{\pi x} \cos(t^2)\,dt$, then $g'(x) =$

(A) $\sin(\pi^2 x^2)$ (B) $\pi x \sin(\pi^2 x^2)$ (C) $\pi x \cos(\pi^2 x^2)$

(D) $\cos(\pi^2 x^2)$ $\boxed{\text{(E)}}$ $\pi\cos(\pi^2 x^2)$

$g'(x) = \frac{d}{dx}\int_\pi^{\pi x} \cos(t^2)\,dt = \cos(\pi x)^2 \cdot \pi \rightarrow \boxed{\pi\cos(\pi^2 x^2)}$

3. $\frac{d}{dx}\int_{\sin x}^{4} \sqrt{1+t^2}\,dt =$

(A) $\sqrt{1+\sin^2 x}$ $\boxed{\text{(B)}}$ $-\cos x\sqrt{1+\sin^2 x}$ (C) $-\sqrt{1+\sin^2 x}$

(D) $\cos x\sqrt{1+\sin^2 x}$ (E) $\sqrt{1+\cos^2 x}$

$\frac{d}{dx}\int_4^{\sin x} -\sqrt{1+t^2}\,dt = -\sqrt{1+(\sin x)^2}\cdot\cos x \rightarrow \boxed{-\cos x\sqrt{1+\sin^2 x}}$

4. If f has two continuous derivatives on [5, 10], then $\int_5^{10} f''(t)\,dt =$

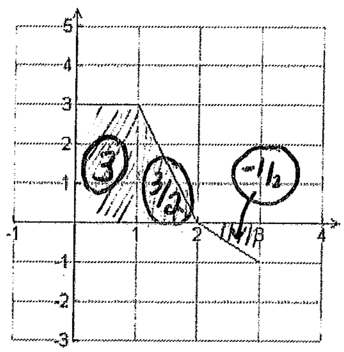
(A) $f'''(10) - f'''(5)$ (B) $f(10) - f(5)$ $\boxed{\text{(C)}}$ $f'(10) - f'(5)$

(D) $f''(10) - f''(5)$ (E) $f''(5) - f''(10)$

$$f'(t)\Big]_5^{10} \rightarrow \boxed{f'(10) - f'(5)}$$

5. The graph of f is given, and g is an antiderivative of f. If $g(3) = 6$, find $g(0)$.

$g'(x)$



$g(x) = \int f(x)\,dx$

$g'(x) = f(x)$

$* \; g(b) = g(a) + \int_a^b g'(x)\,dx$

$g(0) = g(3) + \int_3^0 g'(x)\,dx$

$g(0) = g(3) - \int_0^3 g'(x)\,dx$

$g(0) = 6 - \left(3 + \frac{3}{2} - \frac{1}{2}\right)$

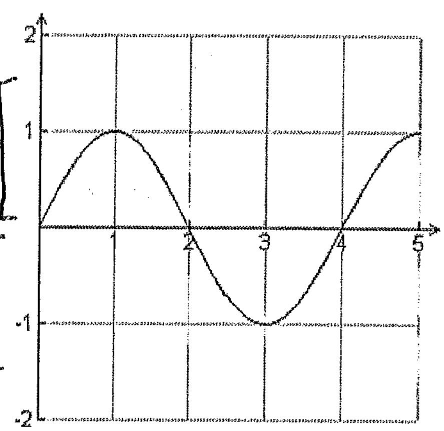
$g(0) = 6 - (4)$

$\boxed{g(0) = 2}$

(A) 1 $\boxed{\text{(B) } 2}$ (C) 4 (D) 5 (E) 10

6. The graph of f is given. $F(x) = \int_0^x f(t)\,dt$

$F'(x)$



$F'(x) = f(x)$

$F'(x)$: ↗ on 0 to 2, ↘ on 2 to 4, ↗ on 4 to 5

$F''(x)$: ∪ on 0 to 1, ∩ on 1 to 3, ∪ on 3 to 5

Which of the following statements is true?

✗ (A) F decreases on (1, 2).

✗ (B) F has a relative minimum at $x = 2$

✓ $\boxed{\text{(C)}}$ F decreases on (2, 4)

✗ (D) F has a relative maximum at $x = 1$.

✗ (E) F has a point of inflection at $x = 4$.

7. $\frac{d}{dx}\int_{x}^{x^2} \tan(t)\,dt =$

(A) $\tan(x^2)-\tan x$ (B) $\tan x-\tan(x^2)$

(C) $\tan x-2x\tan(x^2)$ (D) $2x\tan(x^2)-\tan x$

(E) $\sec^2(x^2)-\sec^2 x$

$\tan(x^2)\cdot 2x - \tan(x)\cdot(1)$

$\rightarrow \boxed{2x\tan(x^2)-\tan(x)}$

8. $\int_1^e \left(x-\frac{5}{x}\right) dx =$

(A) $\frac{1}{2}e^2-\frac{11}{2}$ (B) $\frac{1}{2}e^2-\frac{9}{2}$ (C) $e^2-\frac{11}{2}$

(D) $\frac{1}{2}e^2-\frac{3}{2}$ (E) $\frac{11}{2}-\frac{1}{2}e^2$

$\int x-\frac{5}{x}dx \rightarrow \left.\frac{x^2}{2}-5\ln x\right]_1^e \rightarrow \frac{e^2}{2}-5\ln e-\left(\frac{1}{2}-5\ln(1)\right)$

$\frac{1}{2}e^2-5-\frac{1}{2}-0 \rightarrow$

$\frac{e^2}{2}-\frac{10}{2}-\frac{1}{2}$

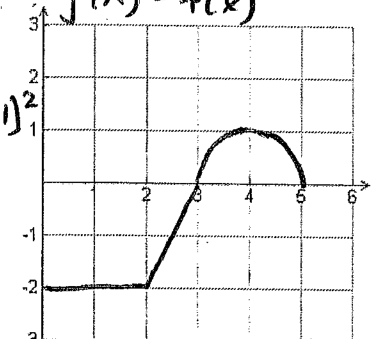
$\frac{e^2-11}{2}$ or

$\boxed{\frac{1}{2}e^2-\frac{11}{2}}$

The graph of f is given. It consists of two line segments and a semi-circle.

$g(x)=\int_1^x f(t)\,dt \rightarrow g'(x)=f(x)$

$g'(x)=f(x)$



(a) Find $g(0)$, $g(1)$, and $g(5)$.

$g(0)=\int_1^0 f(t)dt$

$g(0)=-\int_0^1 f(t)dt=-(-2)=\boxed{2}$

$g(1)=\int_1^1 f(t)dt$

$\boxed{g(1)=0}$

$g(5)=\int_1^5 f(t)dt$

$g(5)=-2-1+\frac{1}{2}(\pi)(1)^2$

$\boxed{g(5)=-3+\frac{\pi}{2}}$

(b) Find $g'(2)$, $g''(2)$, and $g'''(4)$ or state that it does not exist.

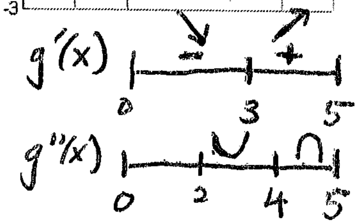
$g'(2)=f(2)$

$\boxed{f(2)=-2}$

$g''(2)=f'(2)$

$\boxed{f'(2)=\text{d.n.e.}}$

$g'''(4)=f''(4)=\boxed{0}$



(c) For what value(s) of x does the graph of g have a point of inflection? Justify your answer.

POI at $x=4$ because $g'(x)$ changes from increasing to decreasing.

(d) Find the absolute maximum and absolute minimum values of g on [0, 5]. Justify your answer.

*EVT: Test relative extrema and endpoints:

$g(0)=2$

$g(3)= \rightarrow \int_1^3 f(t)dt \rightarrow -2-1=-3$

$g(5)=-3+\pi/2 \approx -1.5$

Absolute max value is 2
Absolute minimum value is −3