Calculus **Chapter 6.5a** **U-Substitution Method for Indefinite Integrals**

Key

U-Substitution is the reverse of the derivative chain rule. We want to look for a function and its derivative to be in the integral.

*u-substitution is a method of rewriting an integral problem into a simpler one to help us identify an Integral Rule appropriate for the problem.

Example 1:

Suppose $f(x) = \sin(3x)$

$f'(x) = \cos(3x) \cdot 3$
$f'(x) = 3\cos(3x)$

This means that:

$\int 3\cos(3x)dx = \sin(3x) + C$

U-Substitution Steps:

1. Assign the 'u' value to the expression inside the parentheses.
2. Find the derivative of u: $\frac{du}{dx}$
3. Solve for dx.
4. Rewrite the integral in terms of u and du. (Check to make sure no 'x' or 'dx' remains)
5. Evaluate the Integral
6. Write the answer in terms of x.
7. **Not all Integral problems require U-Substitution. Check first to see if expansion/rewriting problem will allow problem to only need the Power Rule**

Ex. 2: $\int x(x^2+1)^{15}dx$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int x \cdot u^{15} \cdot \frac{du}{2x}$

$\int \cancel{x} \cdot u^{15} \cdot \frac{du}{2\cancel{x}}$

$\frac{1}{2}\int u^{15}du = \frac{1}{2} \cdot \frac{u^{16}}{16} + C$

$= \boxed{\frac{1}{32}(x^2+1)^{16} + C}$

Be sure that variable 'x's cancel out. Remaining constants, coefficients are ok.

Ex. 3: $\int x^2 \sec^2(2x^3)dx$

$u = 2x^3$

$\frac{du}{dx} = 6x^2$

$dx = \frac{du}{6x^2}$

$\int \cancel{x^2} \cdot \sec^2 u \cdot \frac{du}{6\cancel{x^2}}$

$\frac{1}{6}\int \sec^2 u \, du$

$= \frac{1}{6}\tan u + C$

$= \boxed{\frac{1}{6}\tan(2x^3) + C}$

Ex. 4: $\int x^3\sqrt{5-x^4}\,dx = \int x^3(5-x^4)^{1/2}dx$

$u = 5-x^4$

$\frac{du}{dx} = -4x^3$

$dx = \frac{du}{-4x^3}$

$\int \cancel{x^3} \cdot u^{1/2} \cdot \frac{du}{-4\cancel{x^3}}$

$\frac{-1}{4}\int u^{1/2}du$

$= \frac{-1}{4} \cdot \frac{u^{3/2}}{3/2} + C$

$= \boxed{-\frac{1}{6}(5-x^4)^{3/2} + C}$

Ex. 5: $\int \tan^5 x \sec^2 x\,dx$

$$\int (\tan x)^5 (\sec x)^2\,dx$$

$u = \tan x$ | $dx = \frac{du}{\sec^2 x}$

$\frac{du}{dx} = \sec^2 x$

$$\int (u)^5 \cdot \cancel{\sec^2 x} \cdot \frac{du}{\cancel{\sec^2 x}} = \int u^5\,du$$

$$= \frac{u^6}{6} + C = \boxed{\frac{1}{6}\tan^6 x + C}$$

Ex. 6: $\int (3-y)\left(\frac{1}{\sqrt{y}}\right)dy$

$$\int (3-y)(y^{-1/2})\,dy$$

$$\int 3y^{-1/2} - y^{1/2}\,dy$$

$$\frac{3y^{1/2}}{1/2} - \frac{y^{3/2}}{3/2} + C$$

$$\boxed{6y^{1/2} - \frac{2}{3}y^{3/2} + C}$$

Change of Variable U-Substitution Method:

Ex. 7: $\int x\sqrt{x+3}\,dx$

$$\int x(x+3)^{1/2}\,dx$$

$u = x+3$

$\frac{du}{dx} = 1$

$dx = du$

*Creative method of substitution in order to eliminate x-variable

$x = u - 3$

$$\int x \cdot u^{1/2}\,du$$

$$\int (u-3)u^{1/2}\,du$$

$$\int u^{3/2} - 3u^{1/2}\,du$$

$$\frac{u^{5/2}}{5/2} - \frac{3u^{3/2}}{3/2} + C$$

$$\boxed{\frac{2}{5}(x+3)^{5/2} - 2(x+3)^{3/2} + C}$$

Ex. 8: $\int x^2\sqrt{2-x}\,dx$

$$\int x^2(2-x)^{1/2}\,dx$$

$u = 2-x$

$\frac{du}{dx} = -1$

$dx = -du$

$x = 2-u$

$$\int x^2 \cdot u^{1/2} \cdot (-du)$$

$$-\int (2-u)^2 u^{1/2}\,du$$

$$-\int (4-4u+u^2)u^{1/2}\,du$$

$$= \int -4u^{1/2} + 4u^{3/2} - u^{5/2}$$

$$= -\frac{4u^{3/2}}{3/2} + \frac{4u^{5/2}}{5/2} - \frac{u^{7/2}}{7/2} + C$$

$$\boxed{-\frac{8}{3}(2-x)^{3/2} + \frac{8}{5}(2-x)^{5/2} - \frac{2}{7}(2-x)^{7/2} + C}$$

Key

Calculus **Chapter 6.5b** **U-Substitution Method for Definite Integrals**

U-Substitution with definite integrals: Be sure the bounds matches the variable of the problem

Ex. 1: $\int_1^2 2x(x^2-2)^3\,dx$

$u = x^2 - 2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int \cancel{2x} \cdot u^3 \cdot \frac{du}{\cancel{2x}}$

$\int u^3\,du$

convert bounds:

if $x=1$, $u = 1^2 - 2 = -1$

if $x=2$, $u = 2^2 - 2 = 2$

$\int_{-1}^{2} u^3\,du$

$= \frac{u^4}{4}\Big]_{-1}^{2} = \frac{2^4}{4} - \left(\frac{(-1)^4}{4}\right) = \frac{16}{4} - \frac{1}{4}$

$= \boxed{\frac{15}{4}}$

OR:

$\int u^3\,du = \frac{u^4}{4} = \frac{(x^2-2)^4}{4}\Big]_{-1}^{2}$

$= \frac{16}{4} - \frac{1}{4} = \boxed{\frac{15}{4}}$

Ex. 2: $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$

$u = 2x - 1$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$\int \frac{x}{u^{1/2}} \cdot \frac{du}{2}$

* Need to use change of variable method:

$u = 2x - 1$

$\frac{u+1}{2} = x$

$\int \frac{\frac{u+1}{2}}{u^{1/2}} \cdot \frac{du}{2}$

$\frac{1}{4}\int (u+1)u^{-1/2}\,du$

$\frac{1}{4}\int u^{1/2} + u^{-1/2}\,du$

If $x=1$, $u = 2(1) - 1 = 1$

if $x=5$, $u = 2(5) - 1 = 9$

$\frac{1}{4}\frac{u^{3/2}}{3/2} + \frac{1}{4}\frac{u^{1/2}}{1/2}$

$\frac{1}{6}u^{3/2} + \frac{1}{2}u^{1/2}\Big]_1^9 = \frac{1}{6}(9)^{3/2} + \frac{1}{2}(9)^{1/2} - \left(\frac{1}{6} + \frac{1}{2}\right)$

$= \frac{1}{6}(27) + \frac{1}{2}(3) - \frac{1}{6} - \frac{1}{2}$

$= \boxed{\frac{16}{3}}$

OR $\frac{1}{6}u^{3/2} + \frac{1}{2}u^{1/2}$

$= \frac{1}{6}(2x-1)^{3/2} + \frac{1}{2}(2x-1)^{1/2}\Big]_1^5$

$= \frac{1}{6}(9)^{3/2} + \frac{1}{2}(9)^{1/2} - \left(\frac{1}{6} + \frac{1}{2}\right) = \boxed{\frac{16}{3}}$

$$\int_3^{10} f(x)\,dx = -9 \qquad \int_{-1}^{10} f(x)\,dx = \int_{-1}^{3} f(x)\,dx + \int_3^{10} f(x)\,dx$$

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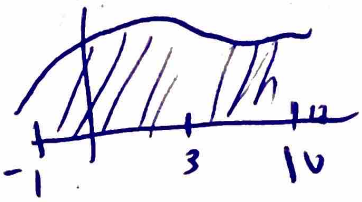
Integrals of Odd and Even Functions

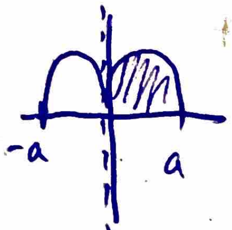
Review: Suppose $\int_{10}^{3} f(x)dx = 9$ and $\int_{-1}^{3} f(x)dx = 5$, find $\int_{-1}^{10} f(x)dx$

$$\int_{-1}^{10} f(x)\,dx = 5 + (-9) = \boxed{-4}$$

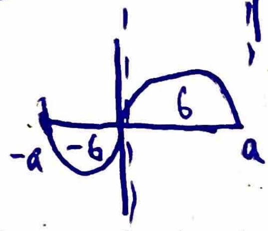


Even/Odd Rules:

Even: $\int_{-a}^{a} f(x)dx = \boxed{2\int_0^a f(x)dx}$



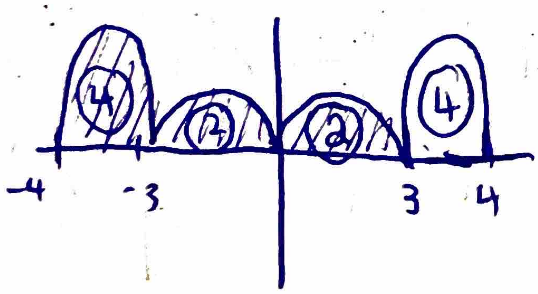
Odd: $\int_{-a}^{a} f(x)dx = 0$



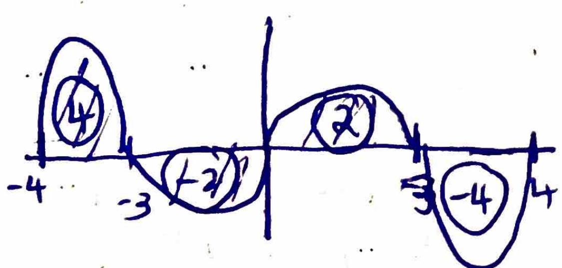
i) start w/ shorter interval
ii) fill out other side
iii) graph larger interval

Ex. 3: Suppose g(x) is an even function where $\int_0^3 g(x)dx = 2$ and $\int_{-4}^{-3} g(x)dx = 4$. Find $\boxed{\int_{-4}^{3} g(x)dx.} = \boxed{8}$

(Sketch a possible graph using the above given information)



Ex. 4: Same as Example 3, but g(x) is an odd function: $\int_0^3 g(x)dx = 2$ and $\int_{-4}^{-3} g(x)dx = 4$. Find $\boxed{\int_{-4}^{3} g(x)dx.} = \boxed{4}$



Ex. 5: If f(x) is even and $\int_3^6 f(x)dx = 7$ and $\int_{-6}^{3} f(x)dx = 12$, find $\boxed{\int_0^6 f(x)dx} = \boxed{9.5}$