6.5 AP Practice Problems (p.453-454)

Key

1. $\int \frac{1}{3}\left(e^x - \frac{2}{x}\right) dx =$

$* \int e^x dx = e^x + c$

$\int \frac{1}{u} du = \ln|u| + c$

(A) $\frac{1}{3}e^x - 2\ln|x| + C$ (B) $\frac{1}{3}\left(e^x - \frac{2}{x^2}\right) + C$

$\boxed{\text{(C)}}$ $\frac{1}{3}e^x - \frac{2}{3}\ln|x| + C$ (D) $\frac{1}{3}(e^x - \ln|2x|) + C$

$\frac{1}{3}\left[e^x - 2\ln|x| + c\right] = \boxed{\frac{1}{3}e^x - \frac{2}{3}\ln|x| + c}$

2. The velocity $v = v(t)$ of an object in rectilinear motion is given by $v(t) = \int t \sin\frac{t}{2}\, dt$. What function gives the acceleration of the object?

$v(t) = \int t\sin\left(\frac{t}{2}\right) dt$

$v'(t) = \frac{d}{dt}\int t\sin\left(\frac{t}{2}\right) dt$

$\boxed{a(t) = t\sin\left(\frac{t}{2}\right)}$

(A) $a(t) = \frac{1}{2}t\cos\frac{t}{2} + \sin\frac{t}{2}$

(B) $a(t) = \frac{1}{2}t\cos\frac{t}{2}$

(C) $a(t) = t^2\cos\frac{t}{2}$

$\boxed{\text{(D)}}$ $a(t) = t\sin\frac{t}{2}$

3. $\int \frac{1 + \sin x}{\cos^2 x} dx =$

$\boxed{\text{(A)}}$ $\tan x + \sec x + C$ (B) $\csc x + \sec^2 x + C$

(C) $x + \sec x + C$ (D) $\sec x + C$

$\int \frac{1}{\cos^2 x} dx + \int \frac{\sin x}{\cos^2 x} dx$

$\int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \rightarrow \int \tan x \sec x\, dx$

$\rightarrow \int \sec^2 x\, dx + \int \tan x \sec x\, dx = \boxed{\tan x + \sec x + C}$

4. Using the substitution $\boxed{u = \sin x}$, $\int_{\pi/6}^{\pi/2} \sin^4 x \cos x\, dx$ becomes

(A) $-\int_{1/2}^{1} u^4 du$ (B) $\int_{\pi/6}^{\pi/2} u^4 du$

(C) $\int_{\sqrt{3}/2}^{0} u^4 du$ $\boxed{\text{(D)}}$ $\int_{1/2}^{1} u^4 du$

$$\int_{\pi/6}^{\pi/2} (\sin x)^4 \cos x\, dx$$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$$\int u^4 \cdot \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}}$$

$$\int u^4 du$$

convert bounds

If $u = \sin x$, $u = \sin(\pi/6) = \frac{1}{2}$

If $u = \sin x$, $u = \sin(\pi/2) = 1$

$$\boxed{\int_{1/2}^{1} u^4 du}$$

5. If the function f is continuous for all real numbers, and if $F'(x) = f(x)$, then $\int_1^4 f(3x)\, dx$ equals

(A) $F(12) - F(3)$ (B) $\frac{1}{3}[F(4) - F(1)]$

$\boxed{\text{(C)}}$ $\frac{1}{3}[F(12) - F(3)]$ (D) $3[F(4) - F(1)]$

* $\int_a^b f(x) dx = F(b) - F(a)$

$u = 3x$

$\frac{du}{dx} = 3$

$dx = \frac{du}{3}$

$$\int f(u) \frac{du}{3}$$

$$\frac{1}{3}\int f(u) du$$

$$\frac{1}{3}F(u) \to \frac{1}{3}F(3x)\Big]_1^4$$

$$\boxed{\frac{1}{3}F(12) - \frac{1}{3}F(3)}$$

6. $\int \frac{e^{-3x}}{2 + e^{-3x}}\, dx =$

(A) $-3\ln(2 + e^{-3x}) + C$ $\boxed{\text{(B)}}$ $-\frac{1}{3}\ln(2 + e^{-3x}) + C$

(C) $2x - \frac{e^{-3x}}{3} + C$ (D) $\frac{1}{3}\ln|-3e^{-3x}| + C$

$$\int \frac{e^{-3x}}{2 + e^{-3x}} dx$$

$u = 2 + e^{-3x}$

$\frac{du}{dx} = e^{-3x} \cdot (-3)$

$dx = \frac{du}{-3e^{-3x}}$

$$\int \frac{\cancel{e^{-3x}}}{u} \cdot \frac{du}{-3\cancel{e^{-3x}}}$$

$$-\frac{1}{3}\int \frac{1}{u} du$$

$$-\frac{1}{3}\ln|u| + c$$

$$\boxed{-\frac{1}{3}\ln|2 + e^{-3x}| + C}$$

7. $\int \frac{x}{1+x^2}\,dx =$

(A) $\arctan x + C$

$\boxed{\text{(B)}}$ $\frac{1}{2}\ln|1+x^2| + C$

(C) $\ln|2(1+x^2)| + C$

(D) $2\ln|1+x^2| + C$

$u = 1+x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int \frac{\cancel{x}}{u}\cdot\frac{du}{2\cancel{x}}$

$\frac{1}{2}\int \frac{1}{u}\,du$

$\frac{1}{2}\ln|u| + C$

$\boxed{\frac{1}{2}\ln|1+x^2| + C}$

8. $\int_2^6 \frac{dx}{\sqrt{2x-3}} =$

(A) 1 $\boxed{\text{(B)}}$ 2 (C) 4 (D) 78

$\int \frac{1}{(2x-3)^{1/2}}\,dx$

$u = 2x-3$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$\int \frac{1}{u^{1/2}}\cdot\frac{du}{2}$

$\frac{1}{2}\int u^{-1/2}\,du$

$\frac{1}{2}\cdot\frac{u^{1/2}}{1/2}$

$u^{1/2} \to (2x-3)^{1/2}\Big]_2^6$

$(12-3)^{1/2} - (4-3)^{1/2}$

$9^{1/2} - 1^{1/2} = 3 - 1 = \boxed{2}$

9. What is the average value of $f(x) = \sin x \cos x$ on the closed interval $\left[0, \frac{\pi}{2}\right]$?

$\boxed{\text{(A)}}$ $\frac{1}{\pi}$ (B) $\frac{2}{\pi}$ (C) $\frac{1}{2}$ (D) $\frac{\pi}{4}$

$* f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

$\frac{1}{\pi/2 - 0}\int_0^{\pi/2} \sin x\cos x\,dx$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$\frac{2}{\pi}\int u\cdot\cancel{\cos x}\cdot\frac{du}{\cancel{\cos x}}$

$\frac{2}{\pi}\cdot\frac{u^2}{2} \to \frac{1}{\pi}(\sin x)^2\Big]_0^{\pi/2}$

$\frac{1}{\pi}(\sin\frac{\pi}{2})^2 - \frac{1}{\pi}(\sin 0) = \frac{1}{\pi}(1)^2 = \boxed{\frac{1}{\pi}}$

10. $\int x^3\sqrt{x^2+3}\,dx =$

(A) $\frac{1}{5}(x^2+3)^{5/2} + (3+x^2)^{3/2} + C$

$\boxed{\text{(B)}}$ $\frac{1}{5}(x^2+3)^{5/2} - (3+x^2)^{3/2} + C$

(C) $\frac{2}{5}(x^2+3)^{5/2} - 2(3+x^2)^{3/2} + C$

(D) $\frac{4}{5}(x^2+3)^{5/2} + 4(3+x^2)^{3/2} + C$

$\int x^3(x^2+3)^{1/2}\,dx$

$u = x^2+3$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int x^3\cdot u^{1/2}\cdot\frac{du}{2x}$

$\frac{1}{2}\int x^2\cdot u^{1/2}\,du$

$*$let $u-3 = x^2$

$\frac{1}{2}\int (u-3)u^{1/2}\,du$

$\frac{1}{2}\int u^{3/2} - 3u^{1/2}\,du$

$\frac{1}{2}\cdot\frac{u^{5/2}}{5/2} - \frac{3}{2}\cdot\frac{u^{3/2}}{3/2} + C$

$\frac{1}{5}u^{5/2} - 1u^{3/2} + C$

$\boxed{= \frac{1}{5}(x^2+3)^{5/2} - 1(x^2+3)^{3/2} + C}$

11. An object in rectilinear motion has acceleration $a(t) = 4t - 5$ at time $t \geq 0$. If the object's initial velocity is 3, at what time t does the object first change direction? linear motion

$v(0)=3$ (A) 0 $\boxed{\text{(B)}}$ 1 (C) $\frac{3}{2}$ (D) 3

$v(t) = \int a(t)\,dt$

$v(t) = \int 4t-5\,dt$

$v(t) = \frac{4t^2}{2} - 5t + c$

$v(t) = 2t^2 - 5t + c$

$v(0) = 2(0)^2 - 5(0) + c$

$3 = c$

$v(t) = 2t^2 - 5t + 3$

$0 = 2t^2 - 5t + 3$

$0 = (2t+1)(t-3)$

$t = -\frac{1}{2},\ t = 3$

$v(t)$: $-$ on $(0, 3)$, $+$ after 3

$\boxed{\text{particle first change direction at } t=3}$

12. $\displaystyle\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx =$

(A) $\frac{1}{2}$ (B) 1 $\boxed{\text{(C)}}$ 2 (D) $2(e^2 - \sqrt{e})$

$\displaystyle\int \frac{1}{x(\ln x)^{1/2}}\,dx$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

$\displaystyle\int \frac{1}{\cancel{x} \cdot u^{1/2}} \cdot \cancel{x}\,du$

$\displaystyle\int u^{-1/2}\,du$

$\frac{u^{1/2}}{1/2}$

$2(\ln x)^{1/2}\Big]_e^{e^4}$

$2(\ln e^4)^{1/2} - 2(\ln e)^{1/2}$

$2(4\cancel{\ln e})^{1/2} - 2(1)^{1/2}$

$2(2) - 2 = \boxed{2}$

13. An object moving along the y-axis has velocity $v(t) = 2\cos\left(2t + \frac{\pi}{2}\right)$, $t \geq 0$. If the object is at $y = 5$ when $t = \pi$, find its position when $t = \frac{\pi}{2}$.

(A) 1 $\boxed{\text{(B)}}$ 3 (C) 4 (D) 5

* final position = initial position + displacement

$x(\pi/2) = x(\pi) + \displaystyle\int_{\pi}^{\pi/2} v(t)\,dt$

$x(\pi/2 = 5 + \displaystyle\int_{\pi}^{\pi/2} 2\cos(2t + \pi/2)\,dt$

$u = 2t + \pi/2$

$\frac{du}{dt} = 2$

$dt = \frac{du}{2}$

$\int \cancel{2}\cos u \cdot \frac{du}{\cancel{2}} \rightarrow \int \cos u\,du$

$\sin u \rightarrow \sin(2t + \pi/2)\Big]_{\pi}^{\pi/2}$

$\sin(\pi + \pi/2) - \sin(2\pi + \pi/2)$

$\sin(\frac{3\pi}{2}) - \sin(\frac{5\pi}{2}) = -1 - (1) = \boxed{-2}$

$(-2) \rightarrow$

$x(\pi/2) = 5 - 2 = \boxed{3}$

14. $\displaystyle\int \frac{4}{100 + x^2}\,dx =$

(A) $\frac{1}{25}\tan^{-1}\frac{x}{10} + C$ (B) $\frac{1}{25}\tan^{-1}(10x) + C$

(C) $\frac{2}{5}\tan^{-1} x + C$ $\boxed{\text{(D)}}$ $\frac{2}{5}\tan^{-1}\frac{x}{10} + C$

$4\displaystyle\int \frac{dx}{(10)^2 + (x)^2} \rightarrow$ * $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C \rightarrow \frac{1}{10}\arctan\left(\frac{x}{10}\right) + C$

$\rightarrow 4 \cdot \frac{1}{10}\arctan\left(\frac{x}{10}\right) + C \rightarrow \boxed{\frac{2}{5}\arctan\left(\frac{x}{10}\right) + C}$