Key

BC Calculus - 6.6 Notes – Integration by Parts

Integration by parts is typically used for the integration of the product of two functions.

* product of 2 different types of functions like logs and polynomials

$$\int f(x)g'(x) = f\cdot g - \int f'g$$

Integration by parts is based on the product rule:

$$[fg]' = f'g + fg' \qquad \frac{d}{dx}[f\cdot g] = f'g + fg'$$

$$\frac{d}{dx}[f\cdot g] - f'g = fg'$$

$$\int \frac{d}{dx}[f\cdot g] - \int f'g = \int fg'$$

$$f\cdot g - \int f'g = \int fg'$$

$$\boxed{\int fg' = fg - \int f'g}$$

Basic rule for choosing f and g':

1. For f: choose something that becomes simpler when you differentiate.
2. For g': choose something that can easily be integrated.

→ priority 1: logs (ex. $\ln x$)
priority 2: algebraic (ex. x^3)

1. $\int x\sin(x)\,dx$

$f = x$ $\qquad g' = \sin x\,dx$

$f' = 1\,dx$ $\qquad g = -\cos x$

$$x\cdot -\cos x - \int -\cos x \cdot 1\,dx$$

$$-x\cos x + \int \cos x\,dx$$

$$\boxed{-x\cos x + \sin x + c}$$

Tabular Integration: Differentiate to 0 for the chosen $f(x)$. Integrate your chosen $g'(x)$ the same number of times. Follow the sign convention, which is plus/minus repeating.

2. $\int x^4 \sin x\,dx$

* choose the algebraic portion as the $f(x) \to x^4$

$f(x)$	sign	$g'(x)$
x^4	+	$\sin x$
$4x^3$	−	$-\cos x$
$12x^2$	+	$-\sin x$
$24x$	−	$\cos x$
24	+	$\sin x$
0		$-\cos x$

$$\boxed{-x^4\cos x + 4x^3\sin x + 12x^2\cos x - 24x\sin x - 24\cos x + c}$$

Practice Problems: $\int fg' = fg - \int f'g$

Integrate the following.

1. $\int x\cos(x)\,dx$

$f = x \quad g' = \cos x\,dx$

$f' = 1\,dx \quad g = \sin x$

$x\sin x - \int \sin x\,dx$

$x\sin x - -\cos x + C$

$\boxed{x\sin x + \cos x + C}$

2. $\int 2x\cos(3x+1)\,dx$

$f = 2x \quad g' = \cos(3x+1)\,dx$

$f' = 2\,dx \quad g = \frac{1}{3}\sin(3x+1)$

$2x\cdot\frac{1}{3}\sin(3x+1) - \int 2\cdot\frac{1}{3}\sin(3x+1)\,dx$

$\boxed{\frac{2}{3}x\sin(3x+1) + \frac{2}{9}\cos(3x+1) + C}$

3. $\int x^2\sin(x)\,dx$

f	g
$+\ x^2$	$\sin x$
$-\ 2x$	$-\cos x$
$+\ 2$	$-\sin x$
$-\ 0$	$\cos x$

$\boxed{-x^2\cos x + 2x\sin x + 2\cos x + C}$

4. $\int 4xe^{3x+1}\,dx$

f	g
$+\ 4x$	e^{3x+1}
$-\ 4$	$\frac{1}{3}e^{3x+1}$
$+\ 0$	$\frac{1}{9}e^{3x+1}$

$\boxed{\frac{4}{3}xe^{3x+1} - \frac{4}{9}e^{3x+1} + C}$

$\int e^{3x+1}\,dx$

$u = 3x+1$

$\frac{du}{dx} = 3$

$dx = \frac{du}{3}$

$\int e^u\cdot\frac{du}{3}$

$\frac{1}{3}\int e^u\,du$

$\frac{1}{3}e^u + C$

$\frac{1}{3}e^{3x+1} + C$

5. $\int_1^{e^2} x^4\ln x\,dx$

*No Tab Method if $f(x) = \ln x$

$f = \ln x \quad g' = x^4\,dx$

$f' = \frac{1}{x}dx \quad g = \frac{x^5}{5}$

$\ln x\cdot\frac{1}{5}x^5 - \int \frac{1}{x}\cdot\frac{x^5}{5}\,dx$

$\frac{x^5}{5}\ln x - \frac{1}{5}\int x^4\,dx$

$\frac{x^5}{5}\ln x - \frac{1}{5}\cdot\frac{x^5}{5}\Big]_1^{e^2} \rightarrow \frac{e^{10}}{5}\ln(e^2) - \frac{e^{10}}{25} - \left(\frac{1}{5}\ln 1 - \frac{1}{25}\right) = \boxed{\frac{2}{5}e^{10} - \frac{e^{10}}{25} + \frac{1}{25}}$

6. $\int \ln x\,dx$

$\int \ln x\cdot 1\,dx$

$f = \ln x \quad g' = 1\,dx$

$f' = \frac{1}{x}dx \quad g = x$

$x\ln x - \int \frac{1}{x}\cdot x\,dx$

$x\ln x - \int 1\,dx$

$\boxed{x\ln x - x + C}$

$$* \int fg' = fg - \int f'g$$

7. $\int_1^2 (3x^2 - 2x + 1)\ln x\,dx$

$f = \ln x \qquad g' = 3x^2 - 2x + 1$

$f' = \frac{1}{x} \qquad g = \frac{3x^3}{3} - \frac{2x^2}{2} + x$

$\ln x\,(x^3 - x^2 + x) - \int \frac{1}{x}(x^3 - x^2 + x)\,dx$

$\ln x\,(x^3 - x^2 + x) - \int x^2 - x + 1\,dx$

$\ln x\,(x^3 - x^2 + x) - \left(\frac{x^3}{3} - \frac{x^2}{2} + x\right)\Big]_1^2$

$\ln 2\,(8 - 4 + 2) - \frac{8}{3} + 2 - 2 - \left(0 - \frac{1}{3} + \frac{1}{2} - 1\right)$

$\boxed{6\ln 2 - 11/6}$

8. $\int x^3 e^x\,dx$

$*$ Tab Method

f	g'
+ x^3	e^x
− $3x^2$	e^x
+ $6x$	e^x
− 6	e^x
+ 0	e^x

$\boxed{x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C}$

9. The table gives values of f, f', g, and g' for selected values of x. If $\int_0^3 f'(x)g(x)\,dx = 6$, then $\int_0^3 f(x)g'(x)\,dx = ?$

x	0	3
$f(x)$	1	5
$f'(x)$	5	−3
$g(x)$	−4	3
$g'(x)$	3	2

$* \int fg' = fg - \int f'g$

$\int_0^3 f(x)g'(x) = fg\Big]_0^3 - \int_0^3 f'g$

$= f(3)g(3) - f(0)g(0) - \int_0^3 f'(g)\,dx$

$5(3) - 1(-4) - 6$

$15 + 4 - 6 = \boxed{13}$

10. Let f be a twice-differentiable function with selected values of f and its derivatives shown in the table. What is the value of $\int_0^3 xf''(x)\,dx$?

x	$f(x)$	$f'(x)$	$f''(x)$
0	2	−2	5
3	5	7	−2

$*$ Tab Method

f	g
+ x	$f''(x)$
− 1	$f'(x)$
+ 0	$f(x)$

$xf'(x) - f(x)\Big]_0^3 = 3f'(3) - f(3) - (0f'(0) - f(0))$

$3(7) - 5 - 0 + 2 = \boxed{18}$

11. $\int x\cos 2x\, dx$

f	g
+ x	$\cos(2x)$
− 1	$\frac{1}{2}\sin(2x)$
+ 0	$-\frac{1}{4}\cos(2x)$

$\int \cos(2x)\, dx$ $\quad u = 2x$ $\quad \frac{du}{dx} = 2$ $\quad dx = \frac{du}{2}$

$\int \cos u \cdot \frac{du}{2}$

$\frac{1}{2}\int \cos u\, du = \frac{1}{2}\sin u \to \frac{1}{2}\sin(2x)$

$\boxed{\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C}$

(A) $\frac{1}{2}x^2\sin(2x) + C$

(B) $\frac{1}{2}x^2\cos(2x) + \frac{1}{2}\sin(2x) + C$

(C) $\frac{1}{2}x\sin(2x) - \frac{1}{4}\cos(2x) + C$

(D) $\boxed{\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C}$

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12. $\int_1^e x^4 \ln x\, dx$

*cannot use Tab Method

$f = \ln x \qquad g' = x^4\, dx$

$f' = \frac{1}{x}dx \qquad g = \frac{x^5}{5}$

$\ln x \cdot \frac{1}{5}x^5 - \int \frac{1}{x}\cdot\frac{x^5}{5}dx$

$\frac{x^5\ln x}{5} - \frac{1}{5}\int x^4 dx$

$\left.\frac{x^5\ln x}{5} - \frac{1}{5}\cdot\frac{x^5}{5}\right]_1^e$

$\frac{e^5\ln e}{5} - \frac{e^5}{25} - \left(0 - \frac{1}{25}\right)$

$\frac{e^5}{5} - \frac{e^5}{25} + \frac{1}{25}$

$\frac{5e^5}{25} - \frac{e^5}{25} + \frac{1}{25} = \boxed{\frac{4e^5+1}{25}}$

A) $\frac{6e^5-1}{25}$ $\qquad$ (B) $\boxed{\frac{4e^5+1}{25}}$ $\qquad$ (C) $\frac{1-e^3}{3}$ $\qquad$ (D) e^4

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13. Let f be a differentiable function such that $\int f(x)\cos x\, dx = f(x)\sin x - \int \frac{1}{2}x^3 \sin x\, dx$. Which of the following could be $f(x)$.

$f = \_\_?\qquad g' = \cos x$

$f' = \frac{1}{2}x^3 \qquad g = \sin x$

$\int f\cdot g' = f\cdot g - \int f'\cdot g$

$\int \frac{1}{2}x^3 = \frac{1}{2}\cdot\frac{x^4}{4} = \boxed{\frac{x^4}{8}}$

A) $\frac{1}{2}\sin x$ $\qquad$ (B) $\frac{1}{2}\cos x$ $\qquad$ (C) $\boxed{\frac{1}{8}x^4}$ $\qquad$ (D) $\frac{1}{2}x^3$