7.08 Operations with Complex Numbers in Polar Form **Date:** ____________________

Find the *product* of two complex numbers in polar form: derive the formula.

For $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 \cdot z_2 =$ $r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$

$$z_1 \cdot z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

Examples: Find the product of the complex numbers in polar form. Answer in both polar form and rectangular form.

1. $z_1 = 4(\cos 225° + i\sin 225°)$ and $z_2 = 3(\cos 90° + i\sin 90°)$

r_1 θ_1 r_2 θ_2

$$z_1 \cdot z_2 = 4\cdot 3\left[\cos(225+90) + i\sin(225+90)\right]$$
$$= 12\left[\cos(315) + i\sin(315)\right] \text{ or } \boxed{12\,\text{cis}(315)}$$

Rectangular Form: $a = r\cos\theta$, $b = r\sin\theta$; $a + bi$

$a = 12\cos 315 \rightarrow 12\left(\frac{\sqrt{2}}{2}\right)$

$b = 12\sin 315 \rightarrow 12\left(-\frac{\sqrt{2}}{2}\right)$

Rectangular Form $\boxed{6\sqrt{2} - 6\sqrt{2}i}$

2. $z_1 = \sqrt{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ and $z_2 = \frac{1}{5}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$$z_1 \cdot z_2 = \sqrt{2}\cdot\frac{1}{5}\left[\cos\left(\frac{2\pi}{3}+\frac{\pi}{6}\right) + i\sin\left(\frac{2\pi}{3}+\frac{\pi}{6}\right)\right]$$

Polar Form

$$= \frac{\sqrt{2}}{5}\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right] \rightarrow \boxed{\frac{\sqrt{2}}{5}\text{cis}\left(\frac{5\pi}{6}\right)}$$

$a = \frac{\sqrt{2}}{5}\cos\left(\frac{5\pi}{6}\right) \rightarrow \frac{\sqrt{2}}{5}\left(-\frac{\sqrt{3}}{2}\right)$

$b = \frac{\sqrt{2}}{5}\sin\left(\frac{5\pi}{6}\right) \rightarrow \frac{\sqrt{2}}{5}\left(\frac{1}{2}\right)$

$\boxed{-\frac{\sqrt{6}}{10} + \frac{\sqrt{2}}{10}i}$

Rectangular Form

Division is the opposite operation from Multiplication. How do you think the pattern changes when we divide two complex numbers in polar form?

For $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 \div z_2 = \frac{z_1}{z_2} =$ $\frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$

$\frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$

Example: Find the quotient of the complex numbers in polar form: $\frac{z_1}{z_2}$. Write the answer in <u>both</u> polar form and rectangular form.

3. $z_1 = 2(\cos 210° + i\sin 210°)$ and $z_2 = 8(\cos 60° + i\sin 60°)$

$\frac{z_1}{z_2} = \frac{2}{8}\left[\cos(210-60) + i\sin(210-60)\right]$

$\frac{1}{4}\left[\cos 150 + i\sin 150\right] \rightarrow \boxed{\frac{1}{4}\,\text{cis}(150)}$ polar

$a = r\cos\theta$ | $a = \frac{1}{4}\cos 150 \rightarrow \frac{1}{4}\left(\frac{-\sqrt{3}}{2}\right)$

$b = r\sin\theta$ | $b = \frac{1}{4}\sin 150 \rightarrow \frac{1}{4}\left(\frac{1}{2}\right)$

$\rightarrow \boxed{-\frac{\sqrt{3}}{8} + \frac{1}{8}i}$ Rectangular

4. $z_1 = \frac{2}{5}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ and $z_2 = \frac{1}{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$\frac{z_1}{z_2} = \frac{2/5}{1/2}\left[\cos\left(\frac{\pi}{2} - \frac{5\pi}{4}\right) + i\sin\left(\frac{\pi}{2} - \frac{5\pi}{4}\right)\right] \rightarrow \frac{2}{5}\cdot\frac{2}{1}\left[\cos\left(\frac{-3}{4}\pi\right) + i\sin\left(\frac{-3\pi}{4}\right)\right]$

$\boxed{\frac{4}{5}\,\text{cis}\left(\frac{-3\pi}{4}\right)}$ polar

$a = \frac{4}{5}\cos\left(\frac{-3\pi}{4}\right) \rightarrow \frac{4}{5}\cos\left(\frac{5\pi}{4}\right) \rightarrow \frac{4}{5}\left(-\frac{\sqrt{2}}{2}\right)$

$b = \frac{4}{5}\sin\left(\frac{-3\pi}{4}\right) \rightarrow \frac{4}{5}\sin\left(\frac{5\pi}{4}\right) \rightarrow \frac{4}{5}\left(\frac{-\sqrt{2}}{2}\right)$

$\boxed{-\frac{2\sqrt{2}}{5} - \frac{2\sqrt{2}}{5}i}$ rectangular

7.08 Practice: Simplify. Express answers in **both** polar form and in rectangular form. Match angle measurement units to the problem, where $0° \le \theta \le 360°$ or $0 \le \theta \le 2\pi$.

1. $6\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\cdot 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

1) $6\cdot 4\left[\cos\left(\frac{\pi}{2}+\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{2}+\frac{\pi}{4}\right)\right]$

$24\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

$\boxed{24\,\text{cis}\left(\frac{3\pi}{4}\right)}$

$a = 24\cos\left(\frac{3\pi}{4}\right) = 24\left(\frac{-\sqrt{2}}{2}\right)$

$b = 24\sin\left(\frac{3\pi}{4}\right) = 24\left(\frac{\sqrt{2}}{2}\right)$

$\boxed{-12\sqrt{2} + 12\sqrt{2}i}$

2. $5(\cos 135° + i\sin 135°)\cdot 2(\cos 45° + i\sin 45°)$

2) $10\left[\cos(135+45) + i\sin(135+45)\right]$

$10\left[\cos 180 + i\sin 180\right] \to \boxed{10\,\text{cis}(180)}$

$a = 10\cos 180 \to 10(-1)$

$b = 10\sin 180 \to 10(0)$

$\boxed{-10 + 0i \text{ or } -10}$

Division

3. $3\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \div \frac{1}{2}(\cos\pi + i\sin\pi)$

3) $\frac{3}{1/2}\left[\cos\left(\frac{3\pi}{4}-\pi\right) + i\sin\left(\frac{3\pi}{4}-\pi\right)\right]$

$3\cdot\frac{2}{1}\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(-\pi/4\right)\right]$

$6\left[\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right]$

$\boxed{6\,\text{cis}\left(\frac{7\pi}{4}\right)}$

$a = 6\cos(7\pi/4) \to 6\left(\frac{\sqrt{2}}{2}\right)$

$b = 6\sin(7\pi/4) \to 6\left(-\frac{\sqrt{2}}{2}\right)$

$\boxed{3\sqrt{2} - 3\sqrt{2}i}$

4. $2(\cos 90° + i\sin 90°)\cdot 2(\cos 270° + i\sin 270°)$

4) $2\cdot 2\left[\cos(90+270) + i\sin(90+270)\right]$

$4\left[\cos 360 + i\sin 360\right]$

$\boxed{4\,\text{cis}\,360}$

$a = 4\cos 360 \to 4(1)$

$b = 4\sin 360 \to 4(0)$

$\boxed{4 + 0i \text{ or } 4}$

5. $3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \div 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

5) $\frac{3}{4}\left[\cos\left(\frac{\pi}{6}-\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6}-\frac{2\pi}{3}\right)\right]$

$\frac{3}{4}\left[\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right]$

$\frac{3}{4}\left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right] \to \boxed{\frac{3}{4}\,\text{cis}\left(\frac{3\pi}{2}\right)}$

$a = \frac{3}{4}\cos\left(\frac{3\pi}{2}\right) \to \frac{3}{4}(0)$

$b = \frac{3}{4}\sin\left(\frac{3\pi}{2}\right) \to \frac{3}{4}(-1)$

$0 - \frac{3}{4}i$

or $\boxed{-\frac{3}{4}i}$

$\frac{9}{4}-\frac{6}{4}=\frac{3}{4}$

6. $4\left(\cos\frac{9\pi}{4}+i\sin\frac{9\pi}{4}\right)\div 2\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

6) $\frac{4}{2}\left[\cos\left(\frac{9\pi}{4}-\frac{3\pi}{2}\right)+i\sin\left(\frac{9\pi}{4}-\frac{3\pi}{2}\right)\right]$

$2\left[\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right] \rightarrow \boxed{2\,cis\left(\frac{3\pi}{4}\right)}$

$a=2\cos\left(\frac{3\pi}{4}\right)\rightarrow 2\left(\frac{-\sqrt{2}}{2}\right)$

$b=2\sin\left(\frac{3\pi}{4}\right)\rightarrow 2\left(\frac{\sqrt{2}}{2}\right)$

$\boxed{-\sqrt{2}+\sqrt{2}i}$

7. $\frac{1}{2}(\cos 60° + i\sin 60°)\cdot 6(\cos 150° + i\sin 150°)$

8. $6\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\div 2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

9. $5(\cos 180° + i\sin 180°)\cdot 2(\cos 135° + i\sin 135°)$

10. $\frac{1}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\div 3\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

6. $4\left(\cos\frac{9\pi}{4}+i\sin\frac{9\pi}{4}\right)\div 2\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

$$\boxed{2\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)}$$

$$2\left(-\frac{\sqrt{2}}{2}\right)+2\left(\frac{\sqrt{2}}{2}\right)i$$

$$\boxed{-\sqrt{2}+\sqrt{2}i}$$

7. $\frac{1}{2}(\cos 60° + i\sin 60°)\cdot 6(\cos 150° + i\sin 150°)$

$$\boxed{3(\cos 210° + i\sin 210°)}$$

$$3\left(-\frac{\sqrt{3}}{2}\right)+3\left(-\frac{1}{2}\right)i$$

$$\boxed{-\frac{3\sqrt{3}}{2} - \frac{3}{2}i}$$

8. $6\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\div 2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$$\boxed{3\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)}$$

$$3(0)+3(1)i$$

$$\boxed{3i}$$

9. $5(\cos 180° + i\sin 180°)\cdot 2(\cos 135° + i\sin 135°)$

$$\boxed{10(\cos 315° + i\sin 315°)}$$

$$10\left(\frac{\sqrt{2}}{2}\right)+10\left(-\frac{\sqrt{2}}{2}\right)i$$

$$\boxed{5\sqrt{2}-5\sqrt{2}i}$$

10. $\frac{1}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\div 3\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$$\boxed{\frac{1}{6}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}$$

$$\frac{1}{6}\left(\frac{\sqrt{3}}{2}\right)+\frac{1}{6}\left(\frac{1}{2}\right)i$$

$$\boxed{\frac{\sqrt{3}}{12}+\frac{1}{12}i}$$