7.09 More Operations with Complex Numbers in Polar Form

Date: ____________

Powers are the shorthand for repeated Multiplication. How do you think the pattern changes when we raise a complex number in polar form to an exponent?

For $z = r(\cos\theta + i\sin\theta)$

$z^n =$ $r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$

$3^5(\sqrt{2})^5 \rightarrow 3^5(\sqrt{2})^4\cdot\sqrt{2} = 972\sqrt{2}$

Examples: Find the power of the complex number in polar form. Answer in both polar form and rectangular form.

$n=5$

1. $z^5 = \left[3\sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\right]^5$

i) $(3\sqrt{2})^5\left[\cos\left(5\cdot\frac{5\pi}{6}\right) + i\sin\left(5\cdot\frac{5\pi}{6}\right)\right]$

$972\sqrt{2}\left[\cos\left(\frac{25\pi}{6}\right) + i\sin\left(\frac{25\pi}{6}\right)\right]$

or $\boxed{972\sqrt{2}\,\text{cis}\left(\frac{25\pi}{6}\right)}$

$972\sqrt{2}\left(\frac{\sqrt{3}}{2}\right) + 972\sqrt{2}\left(\frac{1}{2}\right)i$

$\boxed{486\sqrt{6} + 486\sqrt{2}i}$

Investigation:

Use multiplication (in rectangular form) or the power rule (in polar form): $(-1+\sqrt{3}i)^3$

$-1+\sqrt{3}i$

$r = \sqrt{1^2+\sqrt{3}^2}$

$r = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) \rightarrow -60 + 180 = 120°$

Q2

$z^3 = \left\{2\left[\cos 120 + i\sin 120\right]\right\}^3$

$2^3\left[\cos(120\cdot 3) + i\sin(120\cdot 3)\right]$

$8\left[\cos 360 + i\sin 360\right]$ ← polar form

$8(1) + 8(0)i = \boxed{8}$ ← rectangular form

Again, use either method listed above: $(-1-\sqrt{3}i)^3$

$r = \sqrt{1^2+\sqrt{3}^2} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = 60° + 180$

$\theta = 240°$

$z = 2\left[\cos 240 + i\sin 240\right]$

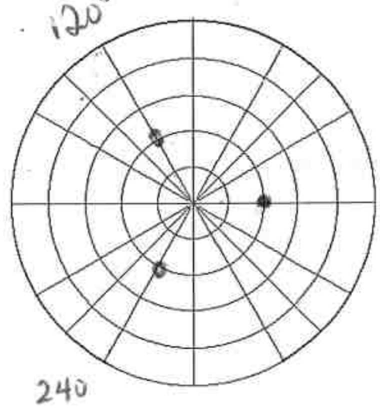
$z = 2^3\left[\cos(240\cdot 3) + i\sin(240\cdot 3)\right]$

$z = 8\left[\cos 720 + i\sin 720\right]$ ← polar

$8 + 0i = \boxed{8}$ ← rectangular

What do you notice?

120°



240

What is $\sqrt[3]{8}$ equivalent to? Plot your answers in the complex plane.

$\sqrt[3]{(-1+\sqrt{3}i)^3} = 8$

$(-1-\sqrt{3}i)^3 = 8$

$\sqrt[3]{8} = 2,\ -1+\sqrt{3}i,\ -1-\sqrt{3}i$

1) $2 + 0i$

2) $2(\cos 120 + i\sin 120)$

3) $2(\cos 240 + i\sin 240)$

Can we use DeMoivre's Theorem (the Power Rule above) to derive a formula for evaluating roots of complex numbers in polar form?

$$\sqrt[n]{z} = z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\tfrac{1}{n}\theta\right) + i\sin\left(\tfrac{1}{n}\theta\right)\right]$$

or

$$r^{\frac{1}{n}}\left[\cos\left(\tfrac{\theta}{n}\right) + i\sin\left(\tfrac{\theta}{n}\right)\right]$$

Example: Find all distinct fourth roots of -5+12*i*.

$r = \sqrt{5^2+12^2} = 13$

$\theta = \tan^{-1}\left(\frac{12}{-5}\right) = -67.38 + 180 = 112.619°$

Q2

① polar form

$13\left[\cos(112.619) + i\sin(112.619)\right]$

② Apply the root $z^{1/4} \rightarrow \sqrt[4]{13}\left[\cos\left(\frac{112.619}{4}\right) + i\sin\left(\frac{112.619}{4}\right)\right]$

i) $\sqrt[4]{13}\left[\cos 28.155° + i\sin 28.155°\right]$

③ Find the other solutions: $\frac{360}{n} \rightarrow n=4 \rightarrow \frac{360}{4} = 90°$

+90

ii) $\sqrt[4]{13}\left[\cos 118.155° + i\sin(118.155°)\right]$

+90

iii) $\sqrt[4]{13}\left[\cos 208.155° + i\sin(208.155)\right]$

+90

iv) $\sqrt[4]{13}\left[\cos 298.155 + i\sin(298.155)\right]$

7. $(-1+\sqrt{3}i)^5$

Convert each complex number into polar form. Then find all distinct roots of the complex number. Answer in polar form where $0 \le \theta \le 2\pi$.

$\boxed{\frac{2\pi}{6} = \frac{\pi}{3}}$ * keep adding $\frac{\pi}{3}$ to get to next solution!

8. Sixth roots of i $\quad 0 + 1i$

$r = 1$

$\theta = \frac{\pi}{2}$

$z = 1\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\pi/2\right)\right]$

$z^{1/6} = \sqrt[6]{1}\left[\cos\left(\frac{1}{6}\cdot\frac{\pi}{2}\right) + i\sin\left(\frac{1}{6}\cdot\frac{\pi}{2}\right)\right]$

i) $z^{1/6} = \sqrt[6]{1}\left[\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right]$

ii) $\sqrt[6]{1}\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right]$

iii) $\sqrt[6]{1}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

iv) $\sqrt[6]{1}\left[\cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right]$

v) $\sqrt[6]{1}\left[\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right]$

vi) $\sqrt[6]{1}\left[\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right]$

9. Fourth roots of $4\sqrt{3} - 4i$

$r = \sqrt{(4\sqrt{3})^2 + (4)^2} = \sqrt{64} = 8$

$\theta = \tan^{-1}\left(\frac{-4}{4\sqrt{3}}\right) = -30 + 360$

$\theta = 330 \cdot \frac{\pi}{180}$

$\theta = \frac{11\pi}{6}$

$z = 8\left[\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right]$

$z^{1/4} = \sqrt[4]{8}\left[\cos\left(\frac{1}{4}\cdot\frac{11\pi}{6}\right) + i\sin\left(\frac{1}{4}\cdot\frac{11\pi}{6}\right)\right]$

i) $\sqrt[4]{8}\left[\cos\left(\frac{11\pi}{24}\right) + i\sin\left(\frac{11\pi}{24}\right)\right]$

$\frac{2\pi}{4} = \boxed{\frac{\pi}{2}}$

ii) $\sqrt[4]{8}\left[\cos\frac{23\pi}{24} + i\sin\left(\frac{23\pi}{24}\right)\right]$

iii) $\sqrt[4]{8}\left[\cos\frac{35\pi}{24} + i\sin\left(\frac{35\pi}{24}\right)\right]$

iv) $\sqrt[4]{8}\left[\cos\left(\frac{47\pi}{24}\right) + i\sin\left(\frac{47\pi}{24}\right)\right]$

10. Fifth roots of unity (1)