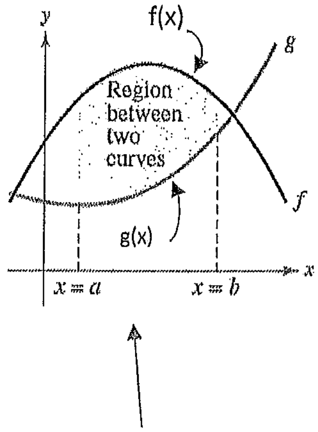
AP Calculus Ch. 7.1 – Area Between Two Curves



Vertical Orientation: (vertical rectangles between graphs)

Right bound → x_2

$$\textbf{\textit{Area}} = \int_{x_1}^{x_2} (Top\ graph - Bottom\ graph)\, dx$$

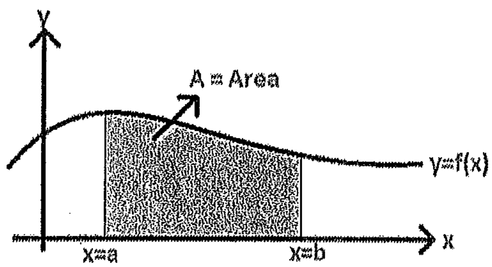
Left bound → x_1

Expressions in terms of **x**

(Equations in the form of " y = ___ ")

Example 1: Area = ______________________________

Example 2:



Area = ______________________________

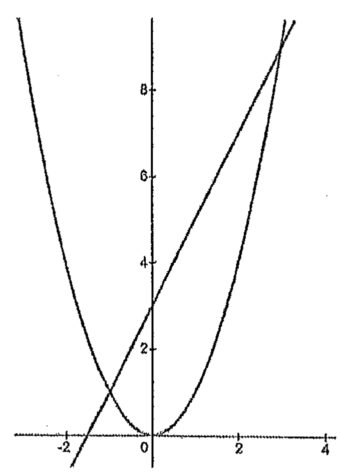
Example 3: Find the area of the region bounded by $y = x^2$ and $y = 2x + 3$

Steps:

i) Find bounds: Find the point of intersection between the 2 graphs (by setting equations equal, & solving for x).

ii) Identify the **top and bottom** function

iii) Apply the Integral **Area Formula.**



②

Horizontal Orientation: (horizontal rectangles between graphs)

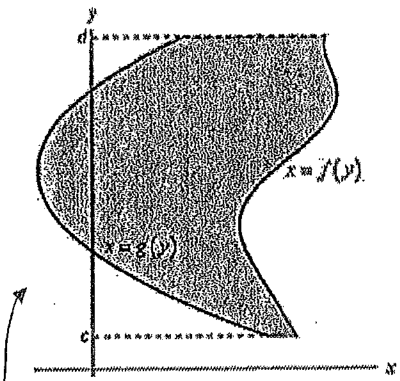
Upper bound

$$Area = \int_{y_1}^{y_2} (Right\ graph - Left\ graph)\, dy$$

Lower bound

Expressions in terms of y

(Equations in the form of " x = ___ ")



Example 3: Area = ____________________

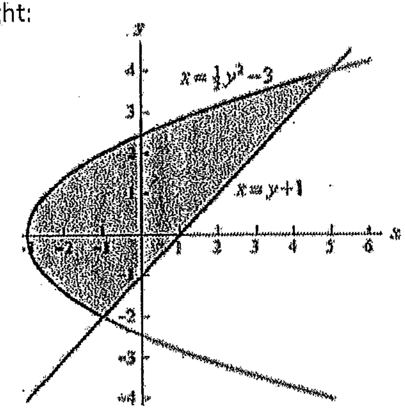
Example 4: Find area of the region bounded by the equations on right:

Steps:

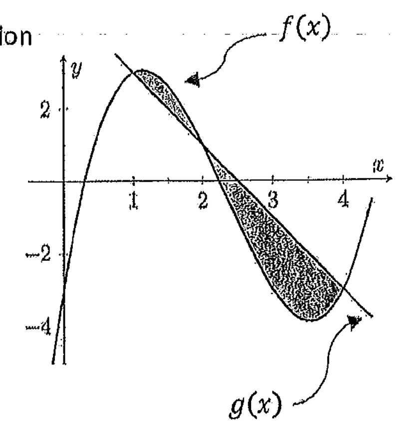
i) Find bounds: Find the point of intersection between the 2 graphs (by setting equations equal, & solving for y).

ii) Identify the **right and left** function

iii) Apply the Integral **Area Formula**



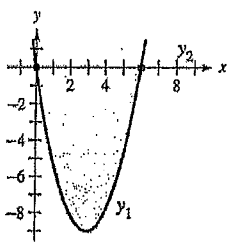
Example 5: Represent the area of shaded region to the right using integral notation



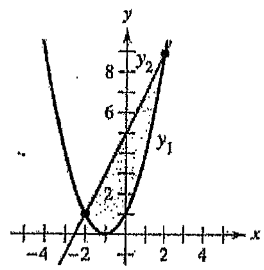
$$Area = \int_{x_1}^{x_2} (Top\ graph - Bottom\ graph)\, dx$$

Writing a Definite Integral In Exercises 1–6, set up the definite integral that gives the area of the region.

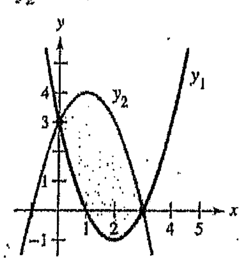
1. $y_1 = x^2 - 6x$
   $y_2 = 0$



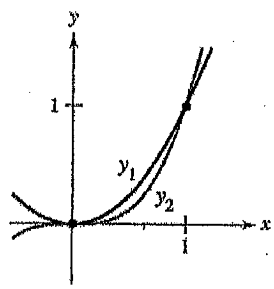
2. $y_1 = x^2 + 2x + 1$
   $y_2 = 2x + 5$



3. $y_1 = x^2 - 4x + 3$
   $y_2 = -x^2 + 2x + 3$



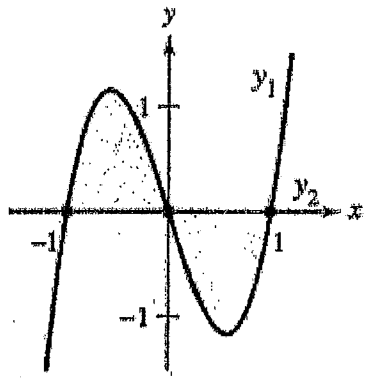
4. $y_1 = x^2$
   $y_2 = x^3$



(4) $Area = \int_{x_1}^{x_2} (Top\ graph - Bottom\ graph)\, dx$

5. $y_1 = 3(x^3 - x)$

$y_2 = 0$



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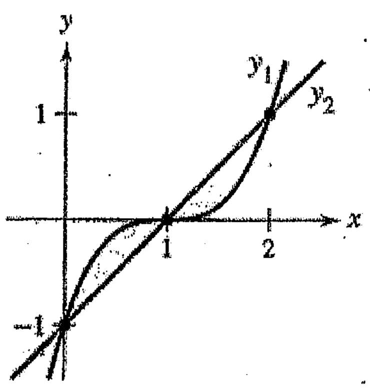
6. $y_1 = (x - 1)^3$

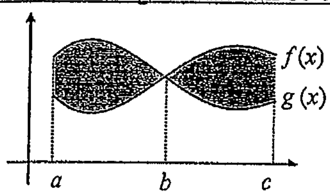
$y_2 = x - 1$



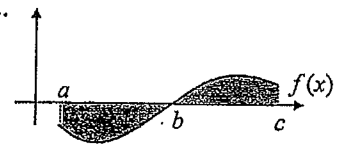
A.P. Calculus AB **7.1 Worksheet** **No Calculators!**

Write an integral that can be used to find the area of the shaded regions.

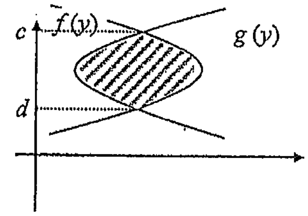
1.



2.



3.



Find the area bounded by the regions listed below:

4. the x-axis and $y = 2x - x^2$

5. the y-axis and $x = y^2 - y^3$

6. $y^2 = x$ and $x = 4$

7. $x = 3y - y^2$ and $x + y = 3$

(6)

8. $y = x^4 - 2x^2$ and $y = 2x^2$

9. $y = x,\ y = \dfrac{1}{x^2},\ x = 2$

10. $4x = y^2 - 4$ and $4x = y + 16$

11. $y = -\sin x$ and $y = 2\sin x$ $, -\pi \le x \le 0$

Ch. 7.1b Area between Curves   Area FRQ Graphing Calculator Practice Problems

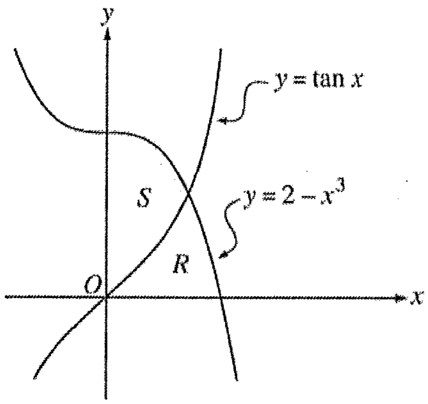
1. Let R and S be the regions in the first quadrant shown in the figure above. The region R is bounded by the x-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$. The region S is bounded by the y-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$.

$$Area = \int_{x_1}^{x_2} (Top\ \ graph - Bottom\ graph)dx \qquad \int_{y_1}^{y_2} (Right\ graph - Left\ graph)dy$$

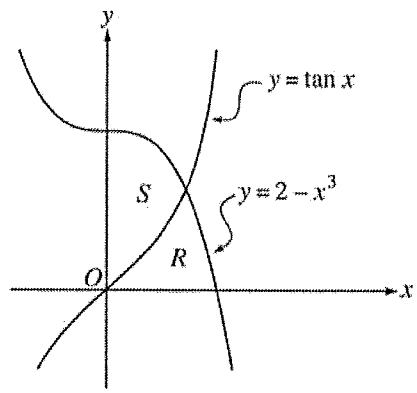
(in the forms of " y = __ ")   (in the form of " x = ___ ")

a) Find the area of **S**

i) (Top – Bottom Method)

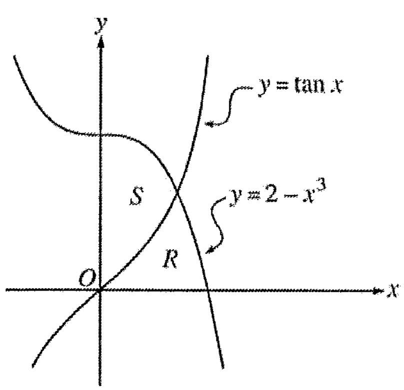


ii) (Right – Left Method)

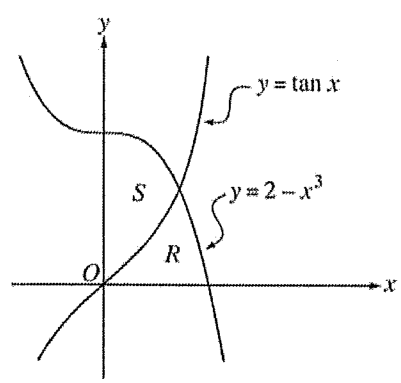


b) Find the area of **R**

i) (Top – Bottom Method)



ii) (Right – Left Method)

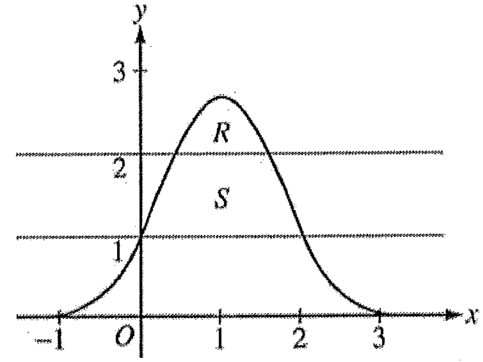


(8)

2) Let R be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal line $y = 2$, and let S be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal lines $y = 1$ and $y = 2$, as shown above.

(a) Find the area of R.

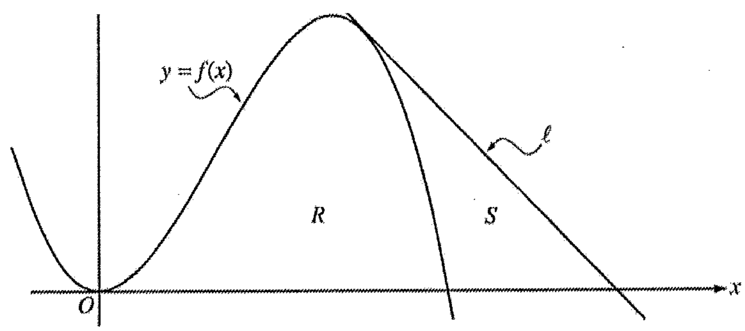
(b) Find the area of S.



3) Let f be the function given by $f(x) = 4x^2 - x^3$, and let ℓ be the line $y = 18 - 3x$, where ℓ is tangent to the graph of f. Let R be the region bounded by the graph of f and the x-axis, and let S be the region bounded by the graph of f, the line ℓ, and the x-axis, as shown above.

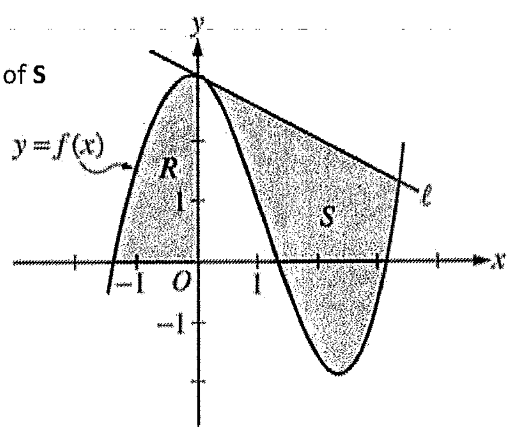
(a) Show that ℓ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

(b) Find the area of S.



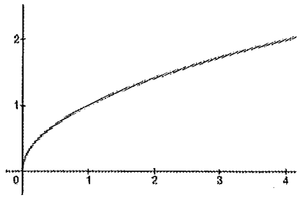
4) Let f be the function given by $f(x) = \dfrac{x^3}{4} - \dfrac{x^2}{3} - \dfrac{x}{2} + 3\cos x$. Let R be the shaded region in the second quadrant bounded by the graph of f, and let S be the shaded region bounded by the graph of f and line ℓ, the line tangent to the graph of f at $x = 0$, as shown above.

(a) Find the area of R. **(b)** Write an <u>integral expression</u> for Area of **S**



Calculus Ch. 7.2a: Volume by Disc Method

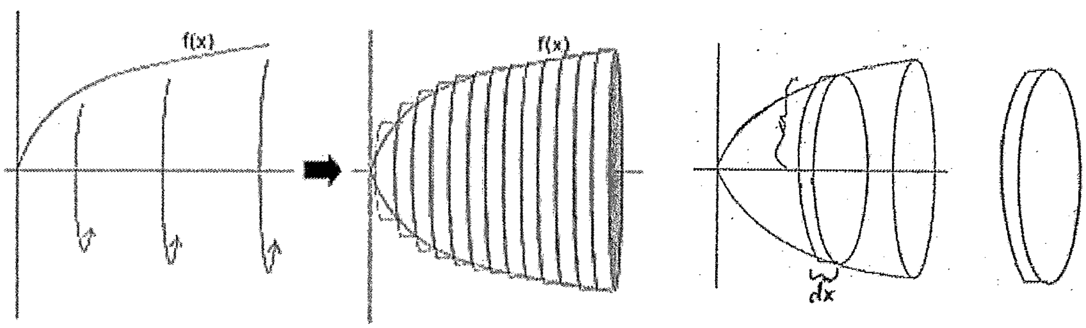
Recall finding area under the curve $y = \sqrt{x}$ between [0, 4]. $Area = \int_a^b (Top\ graph - bottom\ graph)dx$



*Essentially, the Integral Notation allows us to add infinite numbers of differently sized rectangles to form area calculation.

With **Disc Method,** we are going to take this region created by f(x) and the x-axis and rotate this function 360° around the x-axis. What shapes do you see if we were to separate the resulting object into thin slices?

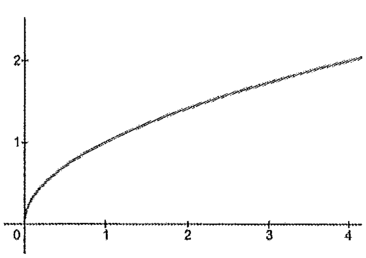
____

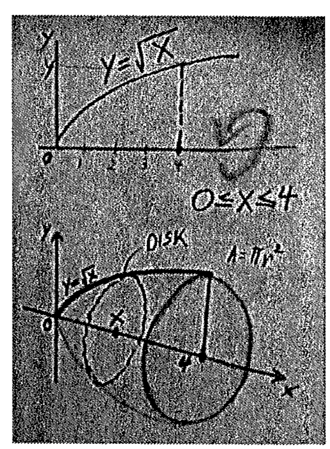


Disc Method: (Top – Bottom) – Vertical Radius	Disc Method: (Right – Left) – Horizontal Radius
$V = \pi \int_{x_1}^{x_2} [R(x)]^2\, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2\, dy$
(expression(s) used above has form: " y = ___")	(expression(s) used above has form: " x = ___")

Radius [R(x) or R(y)] - distance from the AOR (Axis of Revolution) to the **boundary** of shaded region

Example 1: Find the volume of the solid formed by rotating the curve $y = \sqrt{x}$ around the x-axis between [0, 4]





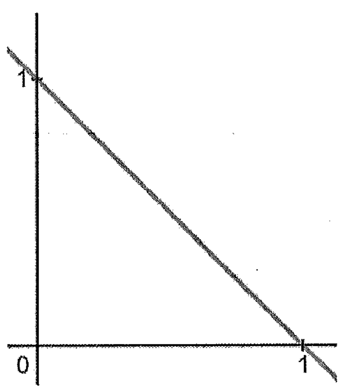
Disc Method: (Top – Bottom) – Vertical Radius – Horizontal AOR	Disc Method: (Right – Left) – Horizontal Radius Vertical AOR
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 \, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 \, dy$
(expression(s) used above has form: " y = ___")	(expression(s) used above has form: " x = ___")

Radius [R(x) or R(y)] : distance from the AOR(Axis of Revolution) to the **outer boundary** of shaded region

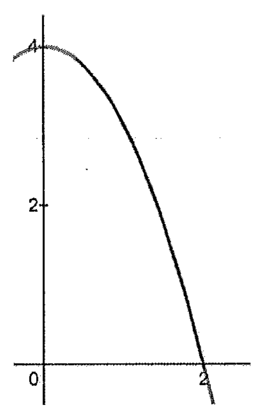
Example 2: Find the volume of the solid created by $f(x) = 2 - x^2$ revolved about the line y = 1.

Example 3: Given the region is formed by the function, x-axis, and y-axis. Find the volume of the solid formed by revolving the region about the **y-axis**

a) $y = -x + 1$



b) $y = 4 - x^2$

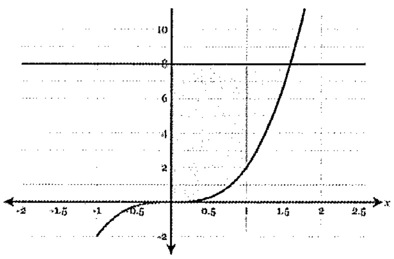


7.2a Disc Method Practice Problems Worksheet

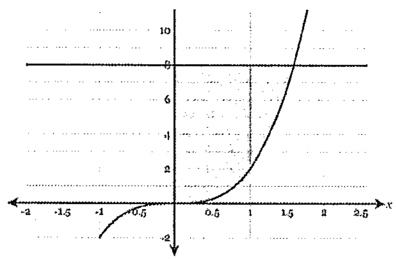
Disc Method: (Top – Bottom)	Disc Method: (Right – Left)
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 \, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 \, dy$
(expression(s) used above has form: " y = ____ ")	(expression(s) used above has form: " x = ____ ")

1. Let the region R be the area enclosed the function $f(x) = 2x^3$ the horizontal line y=8, and the y-axis. Find the volume of the solid generated when the shaded region is:

a) rotated about the line y = 8

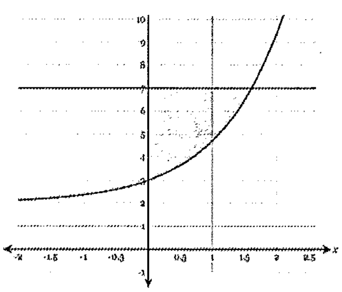


b) rotated about the y-axis

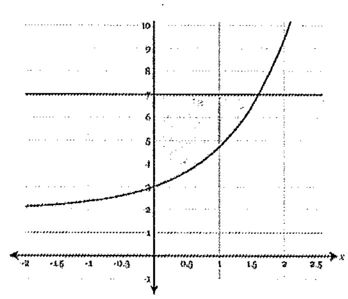


2) Let the region R be the area enclosed the function $f(x) = e^x + 2$, the horizontal line y=7, and the y-axis. Find the volume of the solid generated when the shaded region is:

a) rotated about the line y = 7



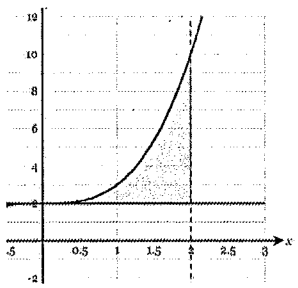
b) rotated about the y-axis



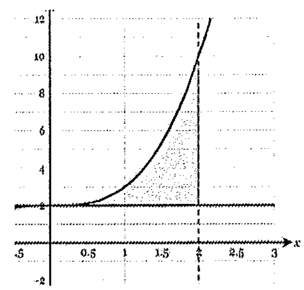
Disc Method: (Top – Bottom)	**Disc Method: (Right – Left)**
$V = \pi \int_{x_1}^{x_2} [R(x)]^2\, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2\, dy$
(expression(s) used above has form: " y = ___")	(expression(s) used above has form: " x = ___")

3) Let the region R be the area enclosed by the function $f(x) = x^3 + 2$, the horizontal line y=2, and the vertical lines x=0 and x=2. Find the volume of the solid generated when shaded region is:

a) rotated about the line y = 2

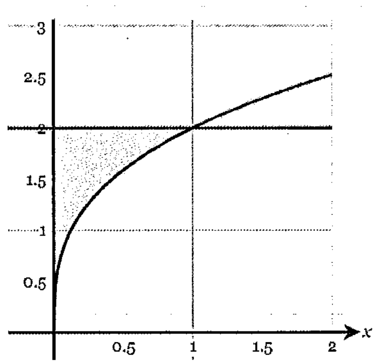


b) rotated about x = 2

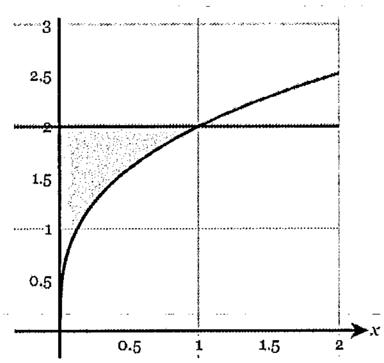


4. Let the region R be the area enclosed the function $f(x) = 2x^{\frac{1}{3}}$, the horizontal line y=2, and the y-axis. Find the volume of the solid generated when shaded region is

a) rotated about the line y = 2

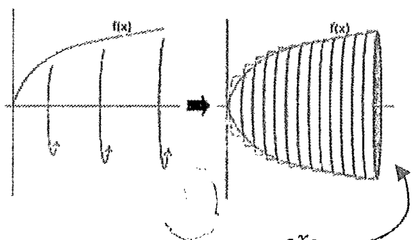


b) rotated about y-axis



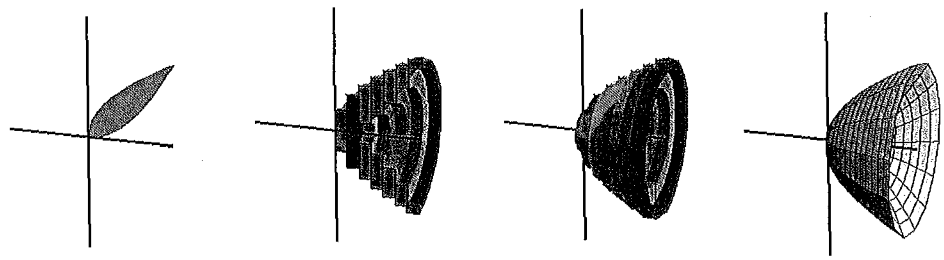
AP Calculus Ch. 7.2b: Volume by Washer Method Notes

Reviewing **Disc Method**



Disc Method: $V = \pi \int_{x_1}^{x_2} [R(x)]dx$

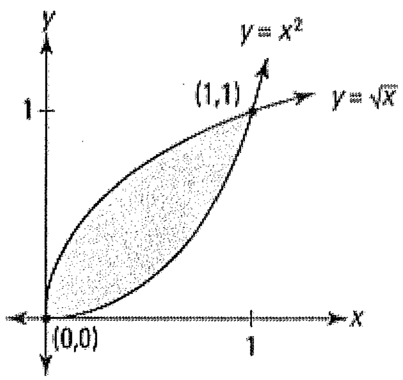
Illustration of **Washer Method**

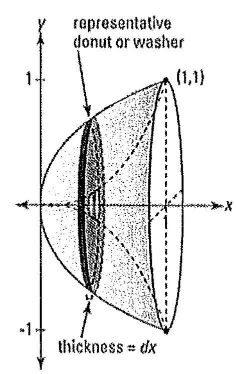


Washer Method: (Top – Bottom) , Vertical Radius (Horizontal AOR)	Washer Method: (Right – Left) , Horizontal Radius (Vertical AOR)
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2\, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$
(expression(s) used above has form: " y = ___")	(expression(s) used above has form: " x = ___")

Radius [R(x) or R(y)] - distance from the AOR (Axis of Revolution) to the **outer**(further)curve
radius[r(x) or r(y)] - distance from the AOR (Axis of Revolution) to the **inner**(closer) curve

Example 1: Find the volume of the solid enclosed by the graphs of $y = x^2$ and $y = \sqrt{x}$, and revolving about the x-axis.





Example 2: Find the volume of the solid created by revolving the function $y = x^2 + 1$ bounded by the line $y = 2$ revolved about the x-axis.

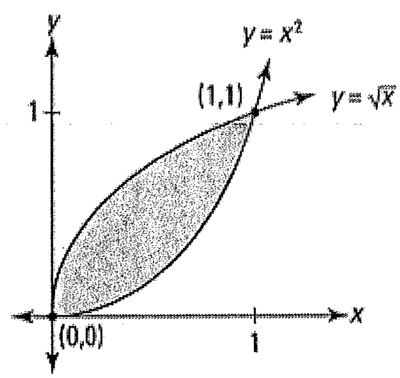
Washer Method: (Top – Bottom), Vertical Radius (Horizontal AOR)	Washer Method: (Right – Left), Horizontal Radius (Vertical AOR)
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2 \, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$
(expression(s) used above has form: " y = ___")	(expression(s) used above has form: " x = ___")

Radius [R(x) or R(y)] - distance from the AOR (Axis of Revolution) to the **outer**(further)curve

radius[r(x) or r(y)] - distance from the AOR (Axis of Revolution) to the **inner**(closer) curve

Example 3: Find the volume of the solid created by revolving the function $y = x^2 + 1$ bounded by the line $y = 2$ and the y-axis about the line $y = 4$

Example 4: Find the volume of the solid created enclosed region of $y = x^2$ and $y = \sqrt{x}$ revolving about the line $x = -2$

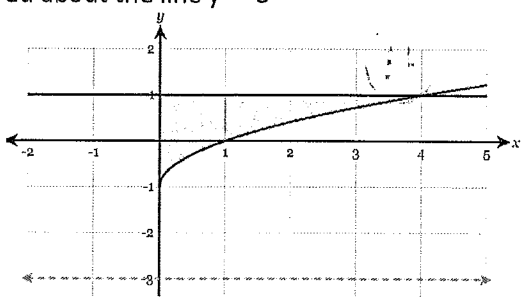


7.2b Volume - Washer Method Practice Problems Worksheet

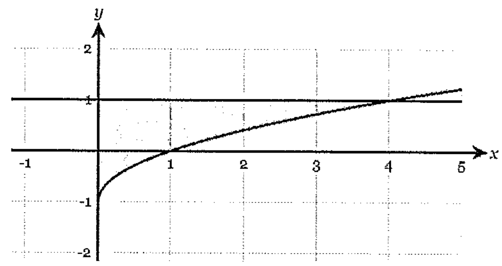
Washer Method: (Top – Bottom) – Vertical Radius	Washer Method: (Right – Left) – Horizontal Radius
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2 \, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$
(expression(s) used above has form: " y = ___")	(expression(s) used above has form: " x = ___")

1. Let the region R be the area enclosed the the function $f(x) = \sqrt{x} - 1$, the horizontal line y=1, and the y-axis. Find the volume of the solid generated when the region is:

a) revolved about the line y = -3

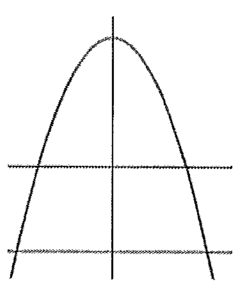


b) revolved about the line x = - 1

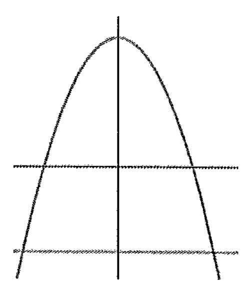


2. Let the region R be the area enclosed the the function $f(x) = 3 - x^2$ the line y = -2. Find the volume of the solid generated when the region is:

a) revolved about the line y = 3



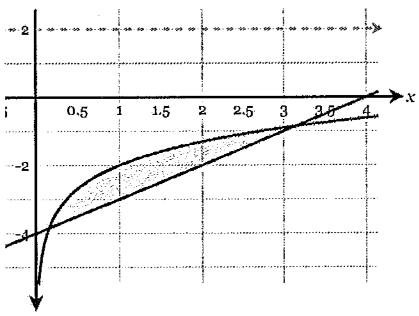
b) revolved about the line y = - 2



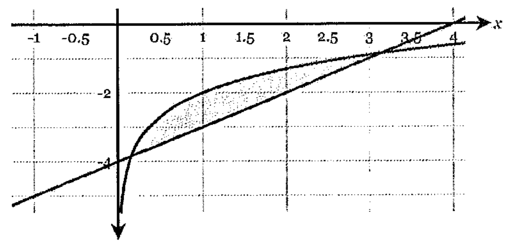
Washer Method: (Top – Bottom) – Vertical Radius	Washer Method: (Right – Left) – Horizontal Radius
$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2 \, dx$	$V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$
(expression(s) used above has form: " y = ___")	(expression(s) used above has form: " x = ___")

3. Let the region R be the area enclosed the function $f(x) = \ln x - 2$ and $g(x) = x - 4$. Find the volume of the solid generated when the region is:

a) revolved about the line $y = 2$

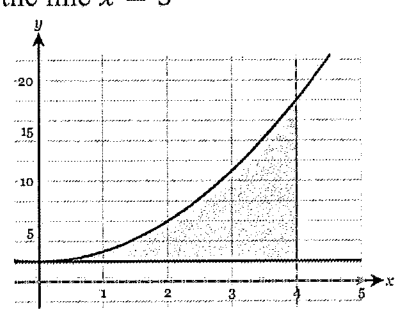


b) revolved about the line x = –1

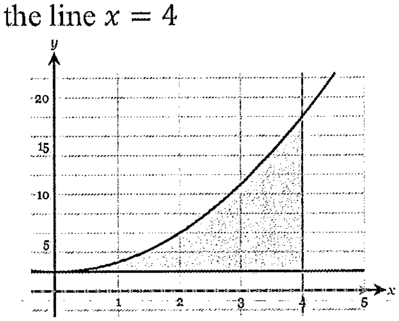


4. Let the region R be the area enclosed by the function $f(x) = x^2 + 2$, the horizontal line y=2, & the vertical lines x=0 & x=4. Find volume of the solid generated when region is:

a) revolved about the line $x = 5$



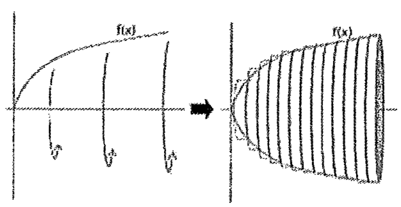
b) revolved about the line $x = 4$



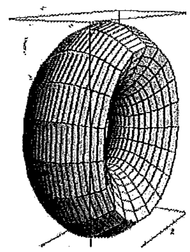
AP Calculus Ch. 7.2c Volumes with Known Cross Section

Cross-section is the shape that results if we cut the object in half and look at the resulting shape sideways:

Disc Method



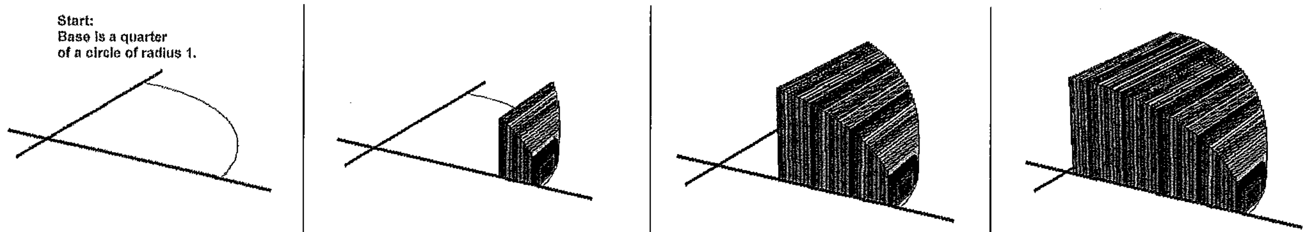
Washer Method



$V = \pi \int [Area \text{ of cross section}]\, dx$

The volume problems we have covered so far(Disc, Washer) have involved taking the shaded region and rotate it about a center, (AOR), resulting in circular Area, either $Area = \pi[R(x)]^2$ or $Area = \pi[R(x)]^2 - \pi[r(x)]^2$

Now, we still have a shaded region, but now it will act as the base of the 3D object. We will now build the cross-section area on top of this base. (no longer rotating around an axis)



Top-Bottom Vertical base	Right-Left Horizontal base
$V = \int_{x_1}^{x_2} [Area \text{ of cross section}]\, dx$ *Note: All values in integral are in terms of x (in the form of " y = ____ ")	$V = \int_{y_1}^{y_2} [Area \text{ of cross section}]\, dy$ *Note: All values in integral are in terms of y (in the forms of " x = ______ ")

Area formulas for Cross sections:

1. Square: $A = (base)^2$	2. Isosceles Right Triangle (**leg** on base): $A = \frac{1}{2}(base)^2$	3. Isosceles Right Triangle (**hypotenuse** on base): $A = \frac{1}{4}(base)^2$
4. Rectangle: A = (base)(height)	5. Equilateral Triangle: $A = \frac{\sqrt{3}}{4}(base)^2$	6. Semicircle: $A = \frac{\pi}{8}(base)^2$

Example 1: Find the volume of the solid if the base is bounded by curve y = $\sqrt{1-x^2}$, y = 0, x = 0 (in the first quadrant) and the cross sections are squares parallel to the y-axis.

Top-Bottom Vertical base	Right-Left Horizontal base
$V = \int_{x_1}^{x_2} [\mathit{Area}\ \text{of cross section}]\,dx$ *Note: All values in integral are in terms of x (equations in the form of " y = ____ ")	$V = \int_{y_1}^{y_2} [\mathit{Area}\ \text{of cross section}]\,dy$ *Note: All values in integral are in terms of y (equations in the form of " x = ____ ")

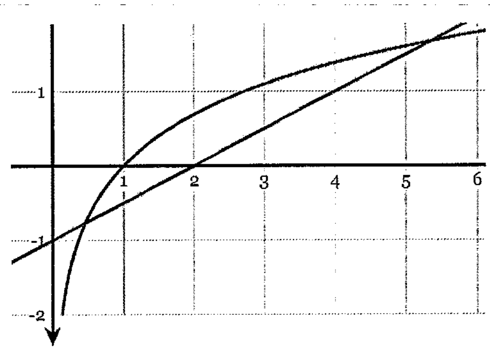
Area formulas for Cross sections:

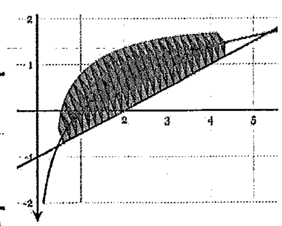
1. Square: $A = (base)^2$	2. Isosceles Right Triangle (leg on base): $A = \frac{1}{2}(base)^2$	3. Isosceles Right Triangle (**hypotenuse** on base): $A = \frac{1}{4}(base)^2$
4. Rectangle: A = (base)(height)	5. Equilateral Triangle: $A = \frac{\sqrt{3}}{4}(base)^2$	6. Semicircle: $A = \frac{\pi}{8}(base)^2$

Example 2: Find the volume of the solid if the base is bounded by the curve $y = x^2$ and the line $y = 4$ and the cross sections are isosceles right triangles whose hypotenuse lie on the base and are parallel to the x-axis.

Example 3: Find the volume of the solid if the base is bounded by the curve $y = x^2$ and the line $y = 4$ and the cross sections are semicircles whose base are perpendicular to the y-axis.

Example 4: Let the region R be the area enclosed by the function $f(x) = \ln x$ and $g(x) = \frac{1}{2}x - 1$. If the region R is the base of a solid such that each cross section perpendicular to the x-axis is an isosceles right triangle with a leg in the region R, find the volume of the solid.





Washer Method: Top 5 Student Mistakes

1) When drawing your radius, **Always include and connect to the Axis of Revolution (AOR, your dotted line)**. Don't connect between the 2 curves. That will not represent the radius. Remember, the AOR is the Center of the rotated object, so the radius needs to connect to the center.

2) Use appropriate Washer Method Formula. Remember to square each of the radius separately.

Correct:

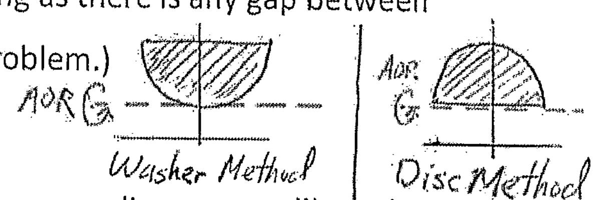
$$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2 dx$$

Incorrect:

$$V = \pi \int_{x_1}^{x_2} [R(x) - r(x)]^2 dx$$

3) The Axis of Revolution has NO impact on the bounds of integration. Your bounds purely depends on the boundaries of your shaded region.

4) Just because the Axis of Revolution is touching the shaded region does not automatically mean this is a Disc Method problem. **Disc Method** is only when the Axis of Revolution is up against **a flat surface** of the shaded region. An Axis up against a **curved surface** means this is **a Washer Method** problem. (As long as there is any gap between AOR and shaded region, this will be a washer method problem.)



5) Remember when you have Horizontal Radius drawn on your diagram, you'll need to adjust your equations so that they start with "x = ____" in order to use them for your radius expressions. The original equations in the form of "y = ____" are suitable for Top-Bottom (Vertical Radius) but not for Right-Left (Horizontal Radius).

6) Remember to distribute the negative through when creating the Radius expressions: Use parentheses to help. Top − (Bottom) or Right − (Left)

7.1-7.2 Area & Volume Formula Sheet

$$\mathbf{Area} = \int_{x_1}^{x_2} (Top\ \ graph - Bottom\ graph)dx$$

(in the forms of " y = __ ")

$$\mathbf{Area} = \int_{y_1}^{y_2} (Right\ graph - Left\ graph)dy$$

(in the form of " x = ___ ")

Disc Method: (Top – Bottom) – Vertical Radius – Horizontal AOR

$$V = \pi \int_{x_1}^{x_2} [R(x)]^2\, dx$$

(expression(s) used above has form: " y = ___ ")

Disc Method: (Right – Left) – Horizontal Radius Vertical AOR

$$V = \pi \int_{y_1}^{y_2} [R(y)]^2\, dy$$

(expression(s) used above has form: " x = ___ ")

Washer Method: (Top – Bottom) , Vertical Radius (Horizontal AOR)

$$V = \pi \int_{x_1}^{x_2} [R(x)]^2 - [r(x)]^2\, dx$$

(expression(s) used above has form: " y = ___ ")

Washer Method: (Right – Left) , Horizontal Radius (Vertical AOR)

$$V = \pi \int_{y_1}^{y_2} [R(y)]^2 - [r(y)]^2 dy$$

(expression(s) used above has form: " x = ___ ")

Top-Bottom Vertical base

$$V = \int_{x_1}^{x_2} [\boldsymbol{Area}\ \textbf{of cross section}]dx$$

*Note: All values in integral are in terms of x
(in the form of " y = ____ ").

Right-Left Horizontal base

$$V = \int_{y_1}^{y_2} [\boldsymbol{Area}\ \textbf{of cross section}]dy$$

*Note: All values in integral are in terms of y
(in the forms of " x = _______ ")

Area formulas for Cross sections:

1. Square: $A = (base)^2$	2. Isosceles Right Triangle (**leg** on base): $A = \frac{1}{2}(base)^2$	3. Isosceles Right Triangle (**hypotenuse** on base): $A = \frac{1}{4}(base)^2$
4. Rectangle: A = (base)(height)	5. Equilateral Triangle: $A = \frac{\sqrt{3}}{4}(base)^2$	6. Semicircle: $A = \frac{\pi}{8}(base)^2$